The Duodecimal Bulletin • Volume 54_z (64_d), Number 1 • Whole Number $↊4_z$ (124_d) • December 1206_z (2022_d)

The Duodecimal Bulletin

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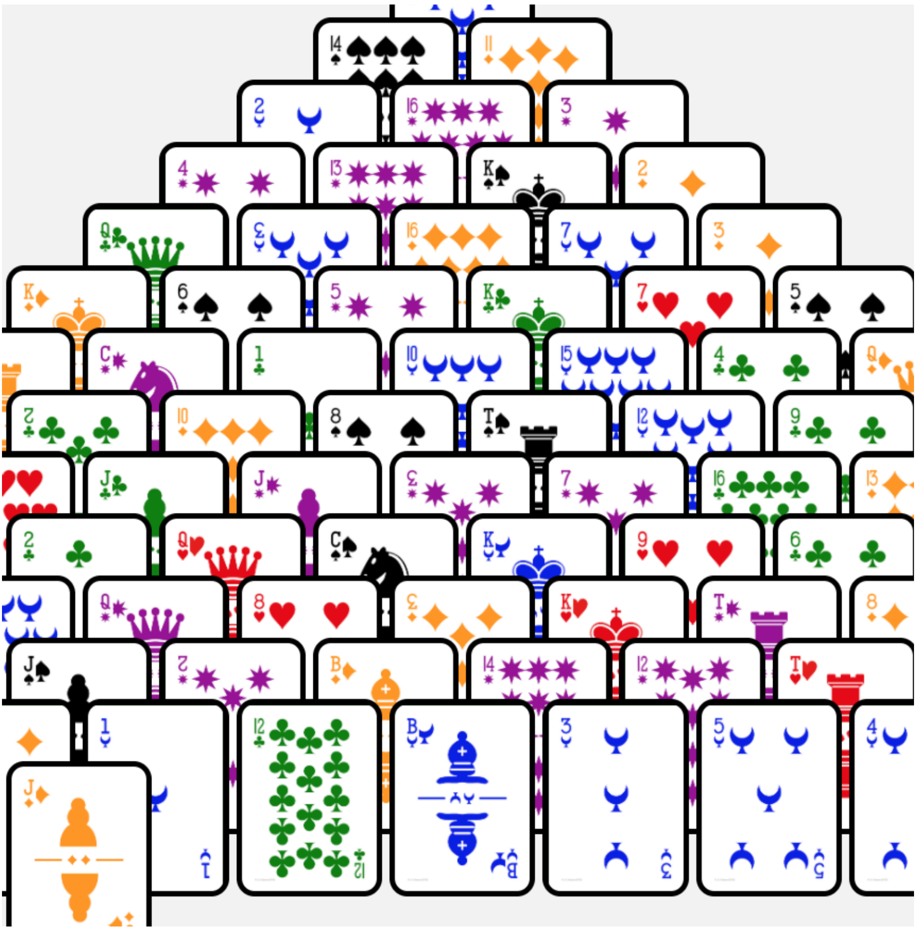

Dozenal Card Games

Whole Number

$↊4_z$

(124_d)

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The Dozenal Society of America

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The Duodecimal Bulletin, Volume Five Dozen Four, Number One, Whole Number Ten Dozen Four.

Volume Five Dozen Four (54_z) • Number 1 • Whole Number Ten Dozen Four ($↊4_z$)

The Duodecimal Bulletin

Volume Sixty Four (64_d) • Number 1 • Whole No. One Hundred Twenty Four (124_d)

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from the Editor's Desk

Is a Turnaround *in the Cards?*

SINCE OUR LAST ISSUE, much in the world has changed. The COVID-03 epidemic[1] has been raging now for three years, forcing people of all walks of life to accept many adjustments we would never have imagined before. Perhaps the disease is attenuating, but it's not clear yet when, if ever, it will be past.

Meanwhile, the conflict devastating the Ukraine has been raging for the better part of a year, with as much of an uncertain outcome as the pandemic. An invasion of one country by another on the European continent, the first since World War II, was heretofore inconceivable in this age of nuclear stalemate. And yet, here it is.

Yet not all news is dire: As I write these words, the unmanned Artemis I spacecraft has just returned from its 21_z day mission to a picture-perfect splashdown, boosting hopes of human beings returning to the Moon for the first time in over four unquennia–and perhaps even going on to Mars within another unquennium.

Much has changed for the dozenal community as well. Donald Goodman III, after many fruitful years at the helm of the Dozenal Society of America, has elected to resign his position as President. We all thank him for his dedicated service and wish him the very best, for himself and his family. As an emeritus member of the Board of Directors, he remains a welcome friend to all in the DSA. Graham Steele, formerly our Vice President, has stepped in to fulfill Don's duties in the interim.

Meanwhile, the difficulties of the pandemic have prevented the DSA from holding an annual in-person meeting. On the other hand, like many during these times, we have learned to adapt by exploiting technology: The DSA Board has managed to meet several times virtually, via Zoom. We are looking into the possibility of using the same means to make the 1207_z Annual Meeting a virtual or hybrid event for the whole membership. We're also considering experimenting with other virtual events, such as presentations or discussion forums on focused topics of dozenal interest, that would be open to all via Zoom, and that could be recorded and made available online on our `dozenal.org` website. To that end, all members who have not already done so are encouraged to join the DozensOnline forum,[2] as that is the best place to receive timely updates about DSA business.

Nevertheless, article submissions to the *Bulletin* remain robust. Paul Rapoport has graced us with no less than three articles detailing his indefatigable efforts to promote fun or useful online applications of dozenal. In fact, the cover article of the issue is his description of online solitaire card games he has developed which utilize dozenal numbers. He also gives us an update on his work to implement a dozenal wristwatch, as well as his project to develop an online dozenal/decimal scientific calculator. The latter has involved contract work from professional programmers, which Paul hired

[1] So called, of course, because it began in the year 1203_z.
[2] `https://www.tapatalk.com/groups/dozensonline`

at his own considerable expense. Although he volunteered this as a contribution to the society, the *Bulletin* would like to appeal to the membership to try out Paul's calculator, and if they find it useful, make whatever donation to the DSA they can and earmark it to defray Paul's cost.

My own contribution to this issue is another installment about the Primel metrology I introduced in the last issue, this time covering the topic of angular mechanics. Perhaps considering how to adapt a coherent dozenal-metric metrology to the physics of objects turning can help us all contemplate how to turn things around in these times.

Professor Lloyd Strickland of Manchester Metropolitan University in Great Britain has discovered a remarkable bit of manuscript by none other than the mathematician and philosopher Gottfried Wilhelm Leibniz. It shows a fascinating glimpse at how Leibniz, more than 230_z years ago, was analyzing the properties of dozenal base. This manuscript has never been published before, but Professor Strickland thought of us and graciously offered the *Bulletin* first dibs at making it public.

Our own Professor Jay Schiffman, DSA Treasurer, offers another of his mathematical morsels, analyzing magic squares, including magic squares containing primes, in both decimal and dozenal base.

One of our younger members, Andon Epp, who came to an Annual Meeting with his parents a few years ago, has written a nice retrospective on the old British pound-shillings-pence currency, complementing it with research on what dozenalists have more recently speculated concerning what a truly dozenal currency might look like. We're encouraged for the future that such thoughtful youths are stepping up to participate. Our recently departed emeritus member Gene Zirkel, longest-serving by far, got his own start in the society writing an article for the *Bulletin* nearly six dozen years ago, when he was not much older than Andon is now.

In fact, Michael de Vlieger, DSA Secretary and former editor of the *Bulletin*, is right now in the process of putting together another issue, to be published shortly after this one, which will be completely devoted as a tribute to Gene Zirkel. Look for that to arrive early in 1207_z.

So whatever is in the cards, let this be my holiday card to you all: Here's wishing that this season find all of our membership well and happy, and enjoying among other things the elegance of dozenal arithmetic, during this dozenth month of 1206_z. However life is treating you, may things turn around to a brighter future in 1207_z!

New Members

Since the last issue, we've seen continuing growth in our membership. Our rolls have gone up by nearly another third.

Joining is easy! You can do it electronically, at the DSA's website: dozenal.org. Just click the **Join Us!** button on the top right. A basic membership is free. For a donation of \$30$_z$ (\$36$_d$) per year, members can subscribe to receive hard copies of the *Bulletin* as they are published. (Subscribing members are highlighted in red below. The electronic version is free to all members.) The DSA Board would like to invite all of our members, new and old, to come to our annual meetings. We'd love to meet you all! Feel free to email editor@dozenal.org your ideas for future *Bulletin* articles. Make sure to visit the DozensOnline Forum at https://www.tapatalk.com/groups/dozensonline/ to chat with us about all subjects related to dozenal and to watch out for announcements to all members.

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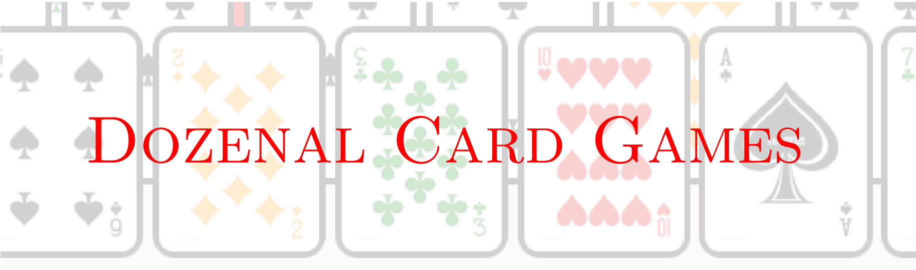


Dozenal Card Games

by Paul Rapoport

What can dozenalists use to attract other people to the dozenal base, without being stymied by the near universal use of decimal? It turns out one endeavor can expose people to dozenal numbers with relative easy while actually entertaining them: dozenal card games. Creating a dozenal deck is easy. In many games, a few adjustments in play or scoring make it possible for players to ignore decimal completely.

The traditional card deck comprises 52_{d} cards: four suits, each with ten numbered "pip cards" and three royal "face cards." The simplest way to dozenalize a deck is to change the label on the ten-pip card from 10_{d} to ↊, then add an eleven-pip card labeled ↋, and a dozen-pip card labeled 10_{z}. Voilà, a fully-dozenal deck of 50_{z} (60_{d}) cards.

Several years ago, the French chess historian Jean-Louis Cazaux created chess games on $12_{\mathrm{d}} \times 12_{\mathrm{d}}$ boards. He furthered his interest in twelve by creating a deck of cards that had pips up to 12_{d}. He named his deck *K6T*. It's available at https://www.thegamecrafter.com/games/k6t.

Cazaux readily agreed to create a fully dozenal deck by changing the cards 10_{d}, 11_{d}, and 12_{d} to ↊, ↋, and 10_{z}. He also expanded his decimal deck in stages to make it even more dozenal: He added two new suits (moons and stars) to make a total of six, and added three new face cards, inspired by chess pieces, to make a total of six in each suit: jack (pawn), cavalier (knight), bishop, tower (rook), queen, king. Then he filled in more pip cards, now up to 24_{d}. The dozenal deck has followed, going up to 20_{z} pips. In the six suits, both decks include a 1 card in addition to the ace, plus a 0 card (intended mostly as a joker/wild card), and an infinity card (whose use has yet to be determined). Two additional blank cards round out the total in either deck to 148_{z} (200_{d}) cards.

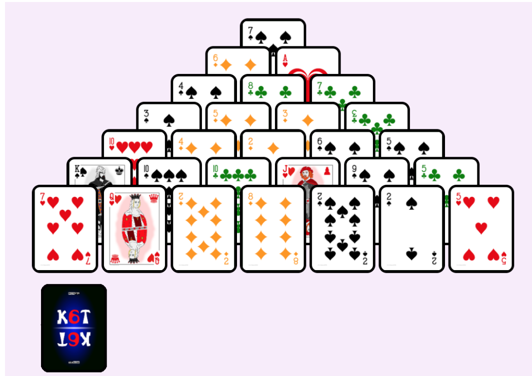
Most card players will wonder why so many cards, when the traditional deck has sufficed for centuries. I suggest that more cards allow the invention of new games that are at least as engaging and fun as traditional games, while offering whole new perspectives to the players. At any rate, players may readily choose how many cards to use for any given game.

In 1203–1204_{z} (2019–2020_{d}), I produced three dozenal solitaire card games for physical play, as well as for online play at https://games.dozenal.ca/solitaire/pyramid/. Their dozenal aspect is vital. The purpose of these games is to teach or reinforce dozenal addition and multiplication in appealing ways.

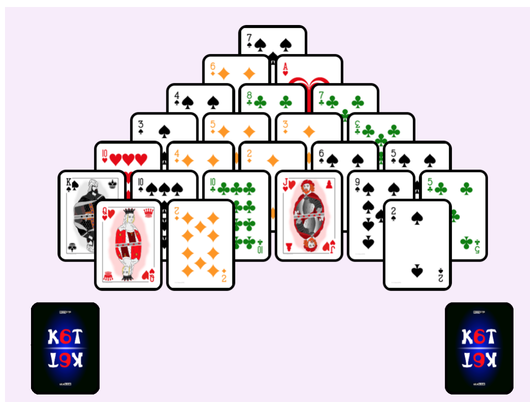
Pyramid Addition

The first game, Pyramid Addition, is similar to the common pyramid solitaire but with a few twists. The dozenal deck size my vary from 40_{z} to 110_{z}. In the smallest, the pips run from A to 9; in the largest, from 0 to 16_z, with both the 1 card and the ace, the ace being the highest card.

The object of the game is simply to empty the tableau by removing pairs of exposed cards that add up to a target. The player chooses the tableau size, a smaller one being much easier to win with than a larger. The example shown here uses the standard dozenal deck (50_z cards) with a small tableau (24_z), and target 13_z. To equal the target, the jack pairs with the 2, the queen with the ace. The king requires no pair, because it is equal to the target and because this configuration does not use the 0 card.



In the above game, the first moves will be to remove the 7 and 8, and the 5 with one of the ↋'s. The ↋ of spades is the better choice, because then the 2 and the jack will be exposed for removal, followed by the remaining ↋ and 5 as shown here.



When no pairs are removable, stock cards are turned over to enable forming a pair with an exposed card in the tableau. Points are totaled at the end of the game, awarded in relation to the size of the deck and tableau, and deducted in relation to how many actions the player has taken to win.

Pyramid Factors

The second game is Pyramid Factors, whose deck is 100_z cards. Although the object of removing pairs is the same, this time the pairs must come from the same group. A "group" consists of numbers related in elementary theory as indicated below, but not already included in a group higher in the list:

- Powers of 2: 2, 4, 8, 14_z
- Multiples of 6: 6, 10_z, 16_z, K(20_z)
- Multiples of the first totative[1]: 5, ↋, 13_z
- Multiples of the second totative: 7, 12_z, B(19_z)
- Multiples of 3 and 4: 3, 9, C(18_z)

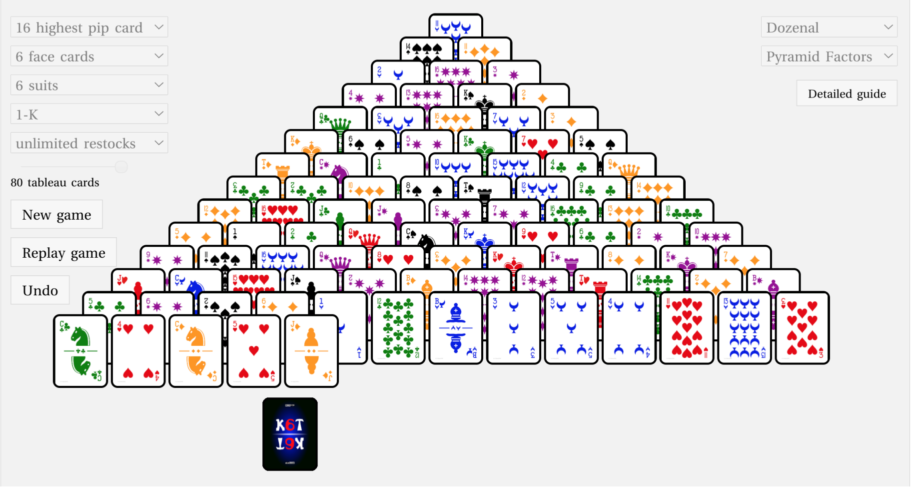
[1] A *totative* has only 1 as a common factor with the base.

- Omega[2], alpha[3], and their multiples: ↋, 11_z, T($1↊_z$)
- Primes: 15_z, J(17_z), Q($1↋_z$)
- One: each 1 is its own group, not paired with another card.

In addition, a pair must not come from adjacent suits.

An exception to coming from the same group is bonus pairs from the first two groups that form a product of 10_z, 20_z, 30_z, 40_z, 60_z, 100_z. Most of these, e.g. 2 × 6 and 8 × 6, have factors from both groups.

Below is a starting tableau of Pyramid Factors with 80_z cards, the maximum. In this game, rather than by portraits, the six face cards are symbolized by their related chess pieces.



Exposed for removal are a C and a 3, a 12_z and a B, a 5 and a 13_z, and an 11_z[4] and an ↋. The stock functions as in Pyramid Addition, and the accumulation of points is similar, with the addition of bonus points for removal of bonus pairs mentioned above.

Hexagon

The game Hexagon is named for the shape of its tableau. It is based on Kings in the Corner solitaire, although it transforms it greatly. The deck is 100_z cards.[5]

The object is to score points by placing cards on the tableau, causing pairs of cards on the same axis (vertical or horizontal) to form a product ending in a factor of a dozen: 2, 3, 4, 6, or 0 (representing 10_z), and optionally 1. So 7 × 9 = 53_z is valid, but 7 × 3 = 19_z is not. All products must not exceed 100_z, precluding 10_z × 11_z, 16_z × 9, etc.

Face cards, which have no numeric value, automatically go from the face-down center pile to their home piles, marked J, C, B, T, Q, and K. One face card goes into

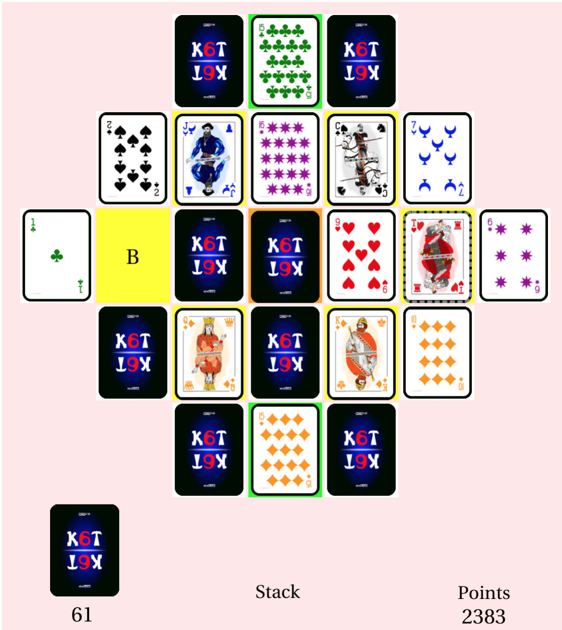
[2] *Omega* means one fewer than the base.

[3] *Alpha* means one more than the base.

[4] The author pronounces 11_z as "anzeen," hence taking the article "an."

[5] A decimal version, Pentagon, exists for 100_d physical cards but is not available as an electronic game.

play at a time, causing any face-down cards needed for the play to turn face up. The player then has three ways to create or improve valid scores on the axes surrounding that face card.



In the tableau above, the T is in play. Both products surrounding it are valid: $7 \times 10_z = 70_z$ and $9 \times 6 = 46_z$, totaling Ɛ6$_z$. Switching the position of the 6 and 10_z results in $7 \times 6 = 36_z$ and $9 \times 10_z = 90_z$, totaling 106_z, a better score by 10_z. Best is to perform the switch and then place a 15_z card on the 7, for $86_z + 90_z = 156_z$.

If no moves result in two valid products surrounding the face card in play, an existing discard may be used, or a card turned over from the stock into the discard pile.

Points accumulate rapidly in Hexagon, but several kinds of error, such as trying to place a card that makes an invalid product, cause them to be deducted as a penalty. A special effect awaits the player who scores 6,000$_z$ points in a game.

Conclusion

Dozenal card games are a viable area of both research and practice, as well as an entertaining tool for teaching and learning arithmetic. Even if not every game is well suited to use in dozenal or with larger decks of cards, the many that are validate both kinds of transformation.

Note: Although I designed the games described, the coding and visual aspects were produced by Thomas Cassidy and Rodrigo Flores. Thanks go also to Jean-Louis Cazaux for making the electronic versions of his K6T cards available.

Time (and Time Again)

by Paul Rapoport

In the previous issue of the *Duodecimal Bulletin* (WN ↊3$_{z}$),[1] I described some possibilities for dozenal timekeeping, including on a wristwatch face. The face that an expert coder produced was for a Pebble watch. The day we finished the software for it was the day Pebble went out of business, in December 1200$_{z}$ (2016$_{d}$).

At that point I decided to find different watch hardware. Having used various online clocks and that watch for several years, I didn't want to give up on a wristwatch with dozenal time.

Sure, wristwatches are hardly *de rigueur* these days, because many people just use their mobile phone to get the time. True dozenal time is hard to find on a mobile, however. That means not just taking a traditional electronic timepiece and doing something simple, like changing the numerals for ten, eleven, and twelve to dozenal, and the four hash marks between numerals to five. To find a timepiece that divides the day by powers of a dozen—starting with the day or half-day—you probably need either one of my online clocks or Uncial Clock Deluxe,[2] created by the *Bulletin*'s editor.

The Bangle.js watch

While my Pebble watch continued to work, the software on it became uneditable. In early 1205$_{z}$, I found the watch Bangle.js (for javascript), which comes from England. Although figuring it out completely was beyond me, I saw immediately that someone could write javascript for it to do what I wanted. Apple watches were not a possibility, because of their requirements and restrictions.

Thomas Cassidy, who did much other work of this sort for me, produced most of the code for that first version of the Bangle watch. When he left the project, David Schonborn finished it. Later in 1205$_{z}$, the Bangle.js version 2 hardware came out, sporting several important improvements. I ported my code to the new watch. The face looked the same, showing the date according to my Holocene calendar (see previous issue), followed by dozenal digital time.[3]

That should have ended the watch project, but it didn't, because I've always wanted a wristwatch in dozenal time that looks analog. Indeed, the very first dozenal timepiece I had someone create, at the University of Illinois, was a large analog clock whose face I clumsily redrew and whose mechanismwas

[1] See https://dozenal.org/drupal/sites_bck/default/files/DuodecimalBulletinIssue531.pdf.

[2] See https://sourceforge.net/projects/uncialclock/files/latest/download.

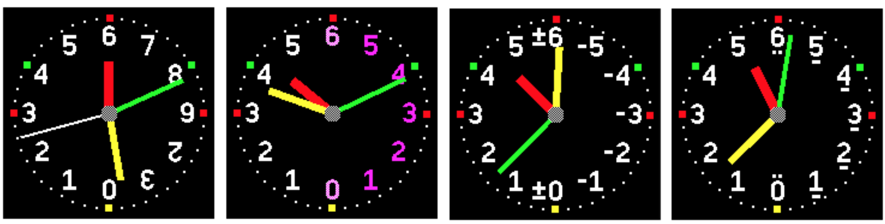
[3] Code for this can be found at https://banglejs.com/apps/?id=doztime.

transformed to run at half speed (with two hands only). I don't know whether that was the first truly dozenal physical clock, but I do know it was created in 1183_z (1971_d).

On the Bangle.js 2, I was able to create the software myself for an imitation analog watch face, because I found a model for traditional time by Andreas Rozek that I could alter and combine with my previous work to produce what I wanted. Prof. Rozek not only gave permission to use his javascript, but for my watch face solved a few problems that I couldn't.

The result was the first face on left below. The four hands, from slowest to fastest, are red, yellow, green, white. The white hand, which revolves once every 42_z (50_d) seconds of traditional time, is optional. Touching the left side of the face toggles to show or hide it. I hide it, which also reduces battery energy output.

Although the lack of a colored square next to the 2 and ↊ is deliberate, I may end up drawing one for each.



Time in Balanced Notation

The three faces to the right all illustrate balanced notation for dozenal time, *balanced* because positive and negative digits are given equal prominence. For arithmetic and measurement, balanced notation was discussed and promoted by J. Halcro Johnston in his 1155_z (1937_d) book *The Reverse Notation*.

With balanced time notation, each digit is understood as modifying the previous digit regardless of signs. Obviously, the time 542_z means 500_z trices after midnight plus 40_z trices after that plus 2 trices after that (equivalent to $10{:}41{:}40_d$). Conversely, the time $5\underline{4}2_z$ means 500_z trices after midnight minus 40_z trices before that plus 2 trices after that (equivalent to $09{:}21{:}40_d$). The time $\underline{5}4\underline{2}_z$ means 500_z trices before midnight plus 40_z trices after that minus 2 trices before that (equivalent to $14{:}38{:}20_d$).

Any time with an initial negative number implies a preceding 1, e.g. $\underline{5}4\underline{2}_z = 1\underline{5}4\underline{2}_z$. But the 1 is usually omitted for convenience.

With a little practice, all the times are understood without any conversion.Whether balanced notation is better than regular is nonetheless a matter for further discussion.

This idea already has some use in traditional clock time. We often say "twenty to" an hour instead of "forty minutes past" the previous hour, e.g. "20_d to 5" instead of "$4{:}40_d$." The *to* expression clearly indicates a subtraction: 5 o'clock minus 20_d minutes.

Expanding and regularizing that possibility lead to a different but consistent way to read all the hands. When a hand passes the midpoint—the 6—it bumps up the reading of the next slower hand by 1 and itself is indicated as negative. In the example on the far right, regular dozenal time, with digits 0 to ↋, would be 516, because each of the hands is at or past its respective numeral. In balanced notation, that time is 52$\underline{6}$. The green hand, now on the negative half of the face, increases the yellow hand's number from 1 to 2. Similarly, on the face to its left, the regular dozenal time would be 461; in balanced notation it looks like 5$\underline{6}$1, although when half a trice goes by, the time becomes 5$\underline{6}$2, because then it's closer to that than to 5$\underline{6}$1.

It may be observed that the number read for a hand position in balanced notation is the numeral the hand is closest to, not always the numeral the hand has passed. The two readings are the same until the hand passes the halfway point between two numerals on the face. In that sense, the time with three hands is rounded digitally to the nearest trice, or with four hands to the nearest lull.

In regular dozenal, the third face from the right would read 438, in balanced notation 44$\underline{4}$.

The three balanced-notation faces are superficial variants of each other. The only difference from regular dozenal notation is that the numerals from 7 to Ɛ are replaced by −5 to −1, with the colored numerals simply representing negatives. I prefer the face with those, because it's less cluttered.

The numerals 6 and 0 are each positive or negative. The 6 is positive for a hand approaching it and negative for a hand having passed it. The 0 has the opposite attributes.

We may also reconfigure the watch face to indicate the day's midnight beginning, uniformly at the bottom here, not with 0 but with −6, and its end, in the same place, with (+)6. That would make noon, at the top, 0. The day would still have 10_z dwells, running from −6 to +6 instead of from 0 to 10_z.

The imitation analog faces make the Holocene calendar date available for a short time by touching their right half, with toggling between cardinal and ordinal date format.

The code for all four faces for the Bangle.js 2 is the same. The user may alter the javascript very slightly to install whichever is wanted. Eventually I may make the choice easier than that. Meanwhile, the code is available to be altered for non-commercial purposes in any way the user prefers.

The Online Dozenal/Decimal Scientific Calculator

by Paul Rapoport

This calculator, produced for the Dozenal Society of America, works in both the dozenal and decimal bases and converts between them. Using mostly the Math.js library, it is web-based and thus independent of computer operating system or browser, working on desktop and mobiles. Its usual maximum precision is 20_z (24_d) fractional digits; for complex numbers it is 12_z (14_d), in each part.

Users may set spacing of numbers to 3 digits (e.g. 123 456.789 ↊↋0 123) or 4 digits (e.g. 1234 5678.9↊↋0 1234). There is a full range of trigonometric functions, using radians, degrees, turns (from Primel[1]), or unciaPis (from TGM[2]). Exponents and logarithms to any base are included. Various number theoretical functions are available, including permutations and combinations with and without repetition, factor count ($\tau(n)$), and Euler totient function ($\phi(n)$).

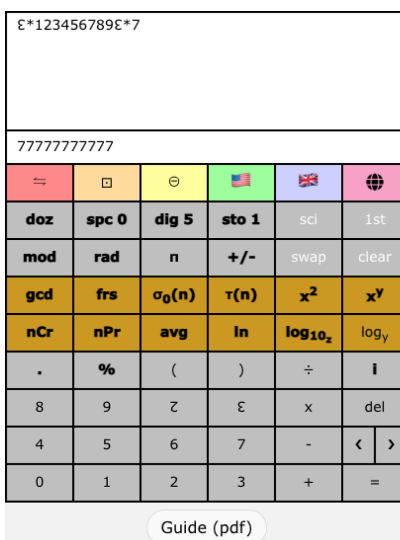
Another unique feature is conversion of measurements among five metrologies: Primel, TGM, USC, Imperial, and Metric. For various measures, e.g. time, velocity, distance, mass, power, electrical current, the calculator converts a measurement from any one metrology to any or all of the others. Measurements from Primel and TGM appear in dozenal, from the others in decimal.

The calculator may be operated at `https://doz-calc.mx-dev.com/`. A button there also brings up the guide to its use.

Comments may be made to `paul@rapoport.ca` or `contact@dozenal.org`. Be advised that this calculator may be revised based on user feedback. Users who would like to help us recoup the cost of development and continued maintenance can make a donation to the Dozenal Society of America. Your contributions would be greatly appreciated.

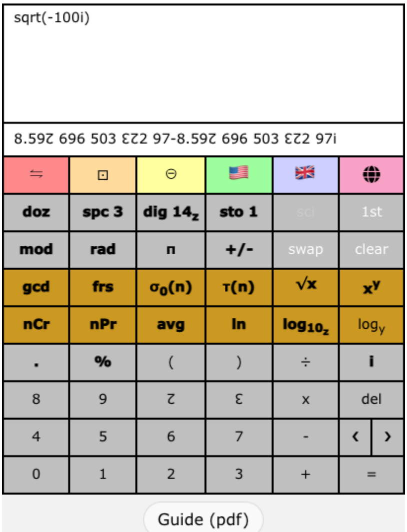
Examples

Example 1. This example demonstrates a well-known phenomenon: In any base $b \geq 3$, where the three highest digits are $\chi = b - 3$, $\psi = b - 2$, $\omega = b - 1$, when a numeral $n = 123\cdots\chi\omega_b$ is formed by concatenating all the digits of b in ascending order but omitting ψ, the product $\omega \times n$ will be a numeral consisting of ω repetitions of digit 1. When multiplied further by any single digit, the result is ω repetitions of that digit. In dozenal, $\chi = 9$, $\psi =$ ↊, $\omega =$ ↋, so e.g. ↋ × 123456789↋$_\mathrm{z}$ × 7 = 77777777777$_\mathrm{z}$.

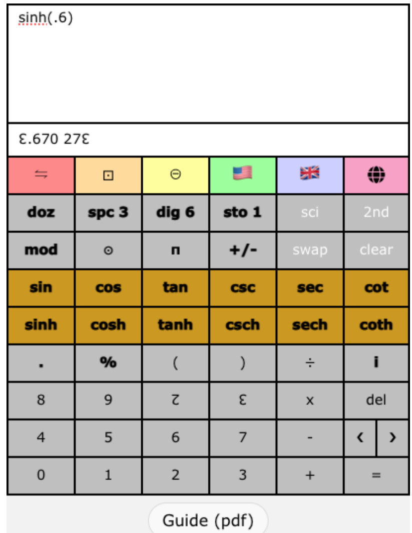


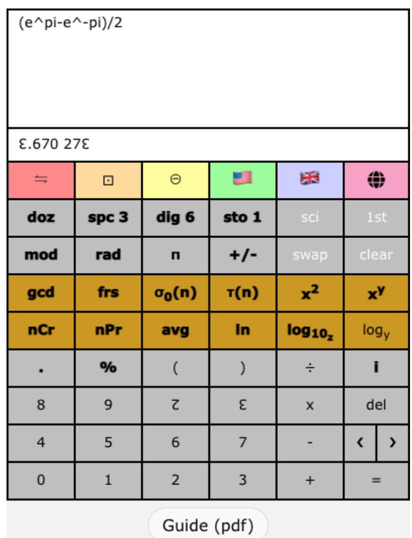
[1] See page 16 or `https://tinyurl.com/ypyew9ke`
[2] `https://tinyurl.com/mv4aznym`

Example 2. This gives the square root of the imaginary number i when multiplied by a negative real number. Although 14_z fractional digits are specified, Math.js produces accuracy only to 12_z for complex numbers, as noted above.

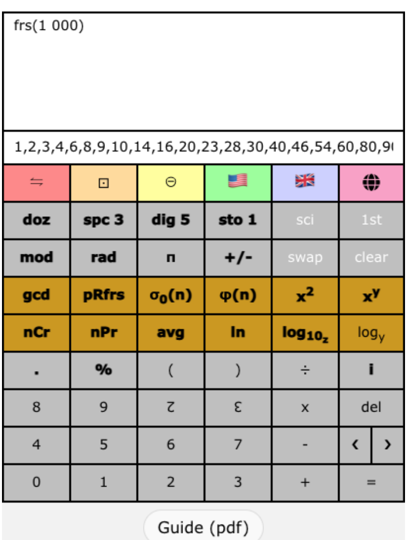


Examples 3 and 4. These demonstrate the identity $\sinh(x) = (e^x - e^{-x})/2$. The equation assumes values in radians. Here for the hyperbolic sine I've specified the angle in Primel turns (0.6_z ⊙) instead. It could also be entered as π radians, 1 Pi (TGM), or 130_z degrees.

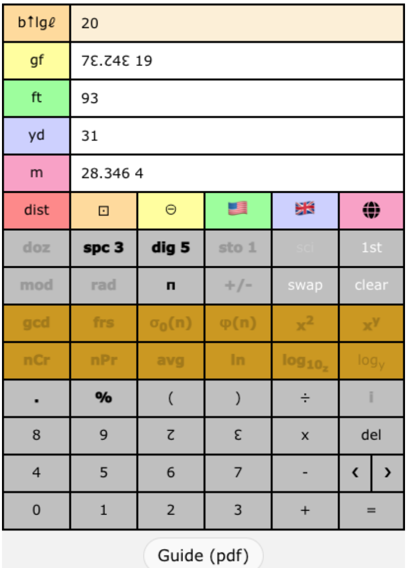


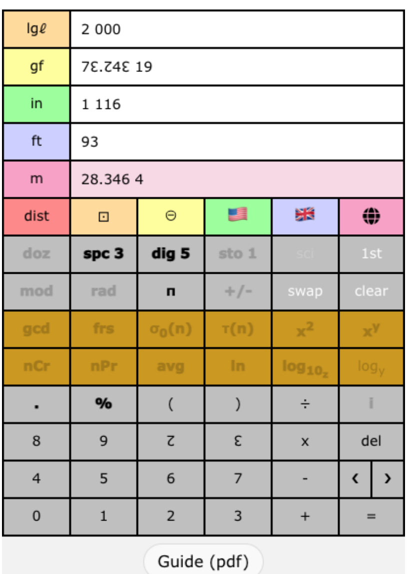


Example 5. This shows the factors of 1000_z. Although they don't all fit into the answer field, scrolling that field reveals the rest of them.



Examples 6 and 7. These show conversions for distance (length, height, etc.) in Primel, TGM, USC, Imperial, and SI. Again, Primel and TGM are in dozenal, the other three in decimal:



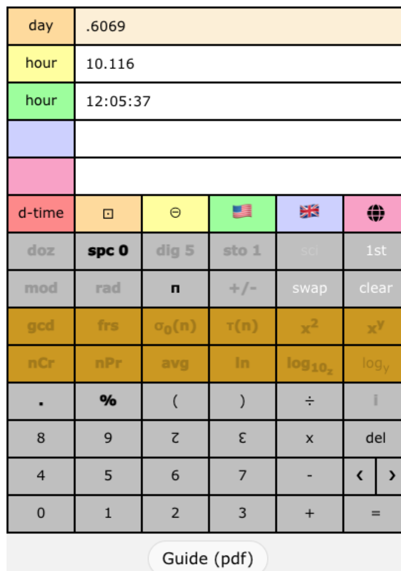


The quantity entered was 20_z ⊡biqua·lengthels; the others then appeared automatically. Then I changed three things: ⊡biqua·lengthels to ⊡lengthels (Primel), feet to inches (USC), and yards to feet (Imperial). Each change in units automatically recalculated the quantity:

20_z ⊡biqua·lengthels → 2000_z ⊡lengthels
93_d feet → $1{,}116_d$ inches
31_d yards → 93_d feet

Although 5 fractional digits are specified, the value in meters needs only 4 for the exact equivalent. The exact foot and yard equivalents require no fractional digits. The Grafut number is rounded.

Example 8. This example converts times of day:



The top number, in Primel, is a dozenal fraction of a day; the next, in TGM, is in dozenal hours (quadquaTims). The equivalent in traditional time, in decimal, is the bottom number. It is actually $12{:}05{:}37.5_d$, although here traditional time's precision is to the most recent whole second, as it is on many time displays.

Primel Metrology, Part II

by John Volan

In the last issue,[1] I presented an introduction to the Primel metrology (brand mark ⊡), a coherent, dozenal-metric, day/gravity/water-based system of measurement which I have been developing for a number of years. I covered the basic units of mechanics and thermodynamics. I also introduced the concepts of "quantitel" unit names, metrology brand prefixes, and "colloquial" unit names.

This article begins covering more advanced topics, including reciprocal units and angular mechanics. Future articles will cover yet more advanced topics.

"Quantitelic" Units Inspired by ISO-31

ISO-31[2] is a standard published by the International Organization for Standardization (ISO).[3] Among many other things, it coined several new English words for reciprocals of certain quantities. This took some terminology that had previously been more ad-hoc, and regularized it:

ISO-31 Coinage	Existing Terminology	Technical Meaning
massic *quantity*	**specific** *quantity*	*quantity* divided by associated mass
volumic *quantity*	[**volumic**] *quantity* **density**	*quantity* divided by associated volume
areic *quantity*	**surface** *quantity* **density**	*quantity* divided by associated area
lineic *quantity*	**linear** *quantity* **density**	*quantity* divided by associated length

This scheme takes a quantity term (such as *mass*) and applies a common *-ic* suffix to turn it into its reciprocal (*massic*), which then can act as a modifier on some other *quantity*. For instance, the previous article replaced the term *specific thermal capacity* with *massic heatability*.

Primel extends this notion to the unit system, turning any unit name into its reciprocal by appending the **-ic** suffix. The **-ic** suffix may be abbreviated with a backslash (\) to indicate that the reciprocal "quantitelic" unit acts as a denominator placed "under" whatever quantity follows it (if any). For instance:

Quantity	Primel Quantitel	Quantitelic Reciprocal	Recip Abbrev	Metric Equivalent
length	⊡**lengthel**	⊡**lengthelic**	⊡*lgℓ*\	$\approx 1.2192024384_{\mathrm{d}}$ per cm
area	⊡**areanel**	⊡**areanelic**	⊡*arℓ*\	$\approx 1.4864545858_{\mathrm{d}}$ per cm^2
volume	⊡**volumel**	⊡**volumelic**	⊡*vmℓ*\	$\approx 1.8122890556_{\mathrm{d}}$ per cm^3
mass	⊡**massel**	⊡**masselic**	⊡*msℓ*\	$\approx 1.8123398011_{\mathrm{d}}$ per gram

[1] https://dozenal.org/drupal/sites_bck/default/files/DuodecimalBulletinIssue531.pdf

[2] https://en.wikipedia.org/wiki/ISO_31, first published in 1992_{d} ($11↊0_{\mathrm{z}}$), superceded by ISO/IEC 80000 (https://en.wikipedia.org/wiki/ISO/IEC_80000) in 2001_{d} ($11↊9_{\mathrm{z}}$).

[3] https://en.wikipedia.org/wiki/International_Organization_for_Standardization

For example, the previous article identified the ⊡masselic·heatabilitel (⊡$m s\ell \backslash h t b \ell$) as Primel's coherent unit of massic heatability. We will see more examples of quantitelic units as we go along.

Primel Units of Angular Mechanics

Angular mechanics (also known as *rotational mechanics*) is the branch of classical mechanics dealing with objects rotating around a fixed axis.

This introduces *plane angle*, or more generally *angular displacement* (symbolized θ), as a distinct type of physical quantity to be measured. For thousands of years, people have been measuring angles using exclusive tools, such as protractors, compasses, and sextants; and have been expressing angle measurements using distinct units not applicable to any other type of quantity. These include **turns** as well as various subdivisions of turns, such as **degrees**, **minutes**, and **seconds** (of arc), although Primel prefers **uncia·turns**, **bicia·turns**, **tricia·turns**, etc., using Systematic Dozenal Nomenclature (SDN) scaling prefixes.[4]

The Radian. As mentioned in the last article, for purposes of physics, the **radian** (abbreviated *rad*) is the most appropriate choice for a coherent unit of angular displacement. This is defined as an angle which subtends a circular arc of length equal to the radius of the enclosing circle. A full turn is τ radians, where $\tau = 2\pi \approx 6.349416967\text{Ɛ}635_{\text{z}}$. So 1 radian is approximately $1.\text{Ƨ}\text{Ɛ}02\text{Ƨ}1443577\text{Ƨ}_{\text{z}}$ uncia·turns or $57.2957795130823_{\text{d}}{}^{\circ}$.

Primel coins **squaradian** (abbreviated *sr*) for the square of the radian. A synonym for this unit, useful in the context of spherical geometry, is the **steradian** (with the same abbreviation). The steradian is the coherent unit of *solid angle* (symbolized Ω). A **spat** is a "full" solid angle covering the entire space surrounding a given vertex. This is σ steradians, where $\sigma = 2\tau = 4\pi \approx 10.696931713\text{Ɛ}06\text{Ƨ}_{\text{z}}$.

Angular Dimensionality. Primel holds that angular displacement is a distinct and sensible physical phenomenon, with an irreducible dimensionality of its own, not commensurate with any other physical quantity. Consequently, the radian constitutes another of Primel's "mundane realities."

Surprisingly, this is a controversial position, because the International System of Units (SI)[5], along with many mainstream mathematicians, consider angles to be *dimensionless* quantities. The radian is actually equated with a pure number, i.e. 1 rad = 1.

This stems from SI's notion that the measure θ of an angle is the ratio of the length s of its subtended arc to the length r of its radius of rotation:

$$\theta = s/r \qquad s = r\theta \qquad r = s/\theta$$

Length over length yields a dimensionless quantity. But this treatment of angles leads to difficulties and inconsistencies across all the quantities of angular mechanics. Radians inexplicably appear and disappear from equations in ad-hoc ways not explicitly driven by the strict algebraic laws of dimensional analysis.

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[4] See page 25.

[5] https://en.wikipedia.org/wiki/International_System_of_Units

This has been so troubling that at least a dozen scientists since 1154_z (1936_d) [6] have written papers advocating for angle to become a first-class dimension, offering various schemes to reconcile the inconsistencies and change the equations of angular mechanics to render them "dimensionally homogeneous."

The approach I will use here is a variant of an idea proposed by Jacques Romain in 1176_z (1962_d).[7] Let the "check" diacritic ($\check{\ }$) act as a kind of operator that may be applied to any quantity q to divide it by 1 radian:

$$\check{q} = \frac{q}{1\ \text{rad}}$$

As a trivial case, if we let $q = 1$ (the pure number one), then $\check{1} = 1\ \text{rad}^{-1}$. Applying ISO-31, we can name this the **radianic** as the reciprocal of the radian.[8] Consequently, we can dub the "check" diacritic the **radianic operator**.

Based on this definition, the radianic operator follows fundamental laws of algebra. Per the associative law, the radianic of a product is the product of either factor multiplied by the radianic of the other; per the distributive law, the radianic of a sum is the sum of the radianics of each of the terms:

$$\widecheck{p \cdot q} = \check{p} \cdot q = p \cdot \check{q} \qquad \widecheck{p + q} = \check{p} + \check{q}$$

So now let an angle θ have first-class physical dimensionality. It retains its intrinsic value no matter what angular units it is measured with. On the other hand, its radianic $\check{\theta}$ is essentially the dimensionless measure quantity when θ is specifically measured in radians: $\theta = \check{\theta}$ rad. So what SI, and most mathematicians, have been working with all along is not θ, but actually $\check{\theta}$:

$$\check{\theta} = s/r \qquad s = r\check{\theta} \qquad r = s/\check{\theta} \qquad \text{(SI)}$$

But Primel endeavors to work only with true dimensioned angles, so the radianic operators must move elsewhere:

$$\theta = s/\check{r} \qquad s = \check{r}\theta \qquad \check{r} = s/\theta \qquad \text{(Primel)}$$

What is the significance of $\check{r}$? It no longer means simply the *radius* of rotation, i.e., the length from the axis of rotation to any point on the circle. Now it means something subtler.

Radiality and the Radiel. I have come to call this new quantity the *radiality* of the rotation (symbolized $\check{r}$). It is the ratio of a length (the radius) to an angle (1 radian); but in fact, as shown in the equation $\check{r} = s/\theta$ above, it is the ratio of any arc length s around the circle, to the angle θ it subtends.

[6] See https://en.wikipedia.org/wiki/Radian#Dimensional_analysis for discussion and references.

[7] Romain, Jacques E. (July 1962_d). "Angle as a fourth fundamental quantity". *Journal of Research of the National Bureau of Standards Section B.* **66B** (3): 97. Freely accessible at https://nvlpubs.nist.gov/nistpubs/jres/66B/jresv66Bn3p97_A1b.pdf. He proposed a bracket notation $\langle q \rangle$ rather than a diacritic.

[8] Some of the solutions to angular dimensionality have involved introducing rad^{-1} as a constant strategically inserted into the equations of angular mechanics. For instance see: Quincey, Paul (1 April 2016_d). "The range of options for handling plane angle and solid angle within a system of units". *Metrologia.* **53** (2): 840–845. Quincey proposes using η to symbolize $\check{1}$. I prefer to reserve $\eta = \tau/4 = \pi/2$ as a circle constant for the right angle.

Radiality characterizes the curvature of the circular path taken by a rotating object. The smaller the radiality, the more angular displacement occurs per linear displacement, and thus the greater the curvature. The larger the radiality, the less angular displacement occurs per linear displacement, and thus the less the curvature.

Primel's coherent unit of radiality is the $\boxdot$**radiel**,[9] abbreviated $\boxdot rd\ell$:

$$\boxdot\textbf{radiel} = \frac{\boxdot\textbf{lengthel}}{\textbf{radian}}$$

This is equivalent to $8.20208\overline{3}_{\mathrm{z}}$ mm·rad^{-1}. An appropriate colloquial synonym for this is $\boxdot$**morsel·radiality**. Scalings of this unit, along with their colloquial synonyms, include:

$\boxdot$**radiel**	$= \boxdot rd\ell$	$= \boxdot$**morsel·radiality**	$= 8.20208\overline{3}_{\mathrm{d}}$ mm·rad^{-1}
$\boxdot$**unqua·radiel**	$= \boxdot u{\uparrow}rd\ell$	$= \boxdot$**hand·radiality**	$= 98.425_{\mathrm{d}}$ mm·rad^{-1}
$\boxdot$**biqua·radiel**	$= \boxdot b{\uparrow}rd\ell$	$= \boxdot$**ell·radiality**	$= 1.1811_{\mathrm{d}}$ m·rad^{-1}
$\boxdot$**triqua·radiel**	$= \boxdot t{\uparrow}rd\ell$	$= \boxdot$**habital·radiality**	$= 14.1732_{\mathrm{d}}$ m·rad^{-1}
$\boxdot$**quadqua·radiel**	$= \boxdot q{\uparrow}rd\ell$	$= \boxdot$**stadial·radiality**	$= 170.0784_{\mathrm{d}}$ m·rad^{-1}
$\boxdot$**pentqua·radiel**	$= \boxdot p{\uparrow}rd\ell$	$= \boxdot$**dromal·radiality**	$= 2.0409408_{\mathrm{d}}$ km·rad^{-1}
$\boxdot$**hexqua·radiel**	$= \boxdot h{\uparrow}rd\ell$	$= \boxdot$**itineral·radiality**	$= 24.4912896_{\mathrm{d}}$ km·rad^{-1}

The reciprocal of the $\boxdot$**radiel** is the $\boxdot$**radielic**, abbreviated $\boxdot rd\ell\backslash$:

$$\boxdot\textbf{radielic} = \frac{\textbf{radian}}{\boxdot\textbf{lengthel}}$$

This is equivalent to approximately $1.219202438404877_{\mathrm{d}}$ rad·cm^{-1}.

The $\boxdot$radiel and the $\boxdot$radielic turn out to be useful modifiers which can neatly transform units of linear mechanics into units of rotational mechanics.

The squares of these will also prove useful. The square of the $\boxdot$**radiel** is the $\boxdot$**squaradiel**, abbreviated $\boxdot sd\ell$:

$$\boxdot\textbf{squaradiel} = \boxdot\textbf{radiel}^2 = \frac{\boxdot\textbf{lengthel}^2}{\textbf{radian}^2} = \frac{\boxdot\textbf{areanel}}{\textbf{steradian}}$$

This is equivalant to $67.2741710069\overline{4}_{\mathrm{d}}$ mm^{2}rad^{-2}. A useful synonym for this, in the context of spherical geometry and solid angles, is the $\boxdot$**steradiel**.

Scalings of this unit, along with their colloquial synonyms, include:

$\boxdot$**squaradiel**	$= \boxdot sd\ell$	$= \boxdot$**morsel·squaradiality**	$= 67.2741710069\overline{4}_{\mathrm{d}}$ mm^2·rad^{-2}
$\boxdot$**biqua·squaradiel**	$= \boxdot b{\uparrow}sd\ell$	$= \boxdot$**hand·squaradiality**	$= 96.87480625_{\mathrm{d}}$ cm^2·rad^{-2}
$\boxdot$**quadqua·squaradiel**	$= \boxdot q{\uparrow}sd\ell$	$= \boxdot$**ell·squaradiality**	$= 1.39499721_{\mathrm{d}}$ m^2·rad^{-2}
$\boxdot$**hexqua·squaradiel**	$= \boxdot h{\uparrow}sd\ell$	$= \boxdot$**habital·squaradiality**	$\approx 2.0087969824_{\mathrm{d}}$ a·rad^{-2}
$\boxdot$**octqua·squaradiel**	$= \boxdot o{\uparrow}sd\ell$	$= \boxdot$**stadial·squaradiality**	$= 2.892666214656_{\mathrm{d}}$ ha·rad^{-2}
$\boxdot$**decqua·squaradiel**	$= \boxdot d{\uparrow}sd\ell$	$= \boxdot$**dromal·squaradiality**	$= 4.16543934910464_{\mathrm{d}}$ km^2·rad^{-2}
$\boxdot$**unnilqua·squaradiel**	$= \boxdot un{\uparrow}sd\ell$	$= \boxdot$**itineral·squaradiality**	$= 599.823266271068_{\mathrm{d}}$ km^2·rad^{-2}

The square of the $\boxdot$**radielic** is the $\boxdot$**squaradielic**, abbreviated $\boxdot sd\ell\backslash$:

$$\boxdot\textbf{squaradielic} = \boxdot\textbf{radielic}^2 = \frac{\textbf{radian}^2}{\boxdot\textbf{lengthel}^2} = \frac{\textbf{steradian}}{\boxdot\textbf{areanel}}$$

[9] Contraction of **radialitel**.

This is equivalent to approximately $1.4864545858124_\mathrm{d}$ rad^{2}cm^{-2}. In the context of spherical geometry, a useful synonym for this unit is ⊡**steradielic**.[↊]

Angular Displacement. The rotational analog for length or linear displacement is of course *angular displacement*. From the equation $\theta = s/\check{r}$ above, we see that angular displacement θ (as defined by Primel) is the ratio of arc length s to the radiality $\check{r}$ of the rotation.

Hence one synonym for Primel's coherent unit of angular displacement is the ⊡**radielic·lengthel**, abbreviated ⊡$rd\ell\backslash lg\ell$:

$$\boxdot\textbf{radielic·lengthel} = \frac{\textbf{radian}}{\boxdot\textbf{lengthel}} \cdot \boxdot\textbf{lengthel} = \textbf{radian}$$

Of course this is just a synonym for the **radian**. Another reasonable synonym is ⊡**ang·lengthel**,[↋] abbreviated ⊡$\measuredangle lg\ell$. (We can generate even more synonyms by substituting synonyms for **lengthel**, including **displacel**, **distancel**, etc.)

Angular Velocity. The rotational analog for linear velocity is *angular velocity* (symbolized ω), which is the time rate of change of angular displacement.

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{\check{r}}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{v_\perp}{\check{r}} \qquad v_\perp = \frac{\mathrm{d}s}{\mathrm{d}t} \qquad \text{(Primel)}$$

where $v_\perp$ is the tangential (linear) velocity. Note that, because SI uses $\check{\theta}$ rather than true θ, and radius r rather than radiality $\check{r}$, its version of angular velocity is $\breve{\omega}$ rather than true ω:

$$\breve{\omega} = \frac{\mathrm{d}\check{\theta}}{\mathrm{d}t} = \frac{1}{r}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{v_\perp}{r} \qquad v_\perp = \frac{\mathrm{d}s}{\mathrm{d}t} \qquad \text{(SI)}$$

This gives $\breve{\omega}$ a problematic dimensionality indistinguishable from that of frequency (inverse time). Similar analysis applies when considering angular velocity as a vector in three dimensions:

$$\boldsymbol{\omega} = \frac{\check{\mathbf{r}} \times \mathbf{v}}{\|\check{\mathbf{r}}\|^2} \qquad \mathbf{v}_\perp = \boldsymbol{\omega} \times \check{\mathbf{r}} \qquad \text{(Primel)}$$

$$\breve{\boldsymbol{\omega}} = \frac{\mathbf{r} \times \mathbf{v}}{\|\mathbf{r}\|^2} \qquad \mathbf{v}_\perp = \breve{\boldsymbol{\omega}} \times \mathbf{r} \qquad \text{(SI)}$$

Primel's coherent unit of angular velocity is the ⊡**radielic·velocitel**, abbreviated ⊡$rd\ell\backslash vc\ell$:

$$\boxdot\textbf{radielic·velocitel} = \frac{\textbf{radian}}{\boxdot\textbf{lengthel}} \cdot \frac{\boxdot\textbf{lengthel}}{\boxdot\textbf{timel}} = \frac{\textbf{radian}}{\boxdot\textbf{timel}}$$

[↊]Tom Pendlebury, in his TGM metrology (see https://dozenal.org/drupal/sites_bck/default/files/tgm_0.pdf), introduced prefixes **rada**, **radi**, **quara**, **quari**, as TGM's own version of (respectively) **radiel**, **radielic**, **squaradiel**, **squaradielic**. Except that Pendlebury's forms resemble his scaling prefixes, rather than independent unit names.

[↋]Contraction of ⊡**angular·lengthel**. I will use **ang** as a contraction for **angular** in synonyms for all the quantitels of angular mechanics.

A reasonable synonym for this is ⊡**ang·velocitel**, abbreviated ⊡ $\measuredangle$*vcℓ*. This is equivalent to 34.56_{d} rad·s^{-1}. Scalings of this unit include:

⊡**unqua·ang·velocitel**	= ⊡$u\uparrow\measuredangle vc\ell$	= 10_{z} rad·vibe^{-1}	= 414.72_{d} rad·s^{-1}
⊡**ang·velocitel**	= ⊡ $\measuredangle vc\ell$	= 1 rad·vibe^{-1}	= 34.56_{d} rad·s^{-1}
⊡**uncia·ang·velocitel**	= ⊡$u\downarrow\measuredangle vc\ell$	= 1 rad·twinkling^{-1}	= 2.88_{d} rad·s^{-1}
⊡**bicia·ang·velocitel**	= ⊡$b\downarrow\measuredangle vc\ell$	= 1 rad·lull^{-1}	= 14.4_{d} rad·min^{-1}
⊡**tricia·ang·velocitel**	= ⊡$t\downarrow\measuredangle vc\ell$	= 1 rad·trice^{-1}	= 1.2_{d} rad·min^{-1}
⊡**quadcia·ang·velocitel**	= ⊡$q\downarrow\measuredangle vc\ell$	= 1 rad·breather^{-1}	= 6_{d} rad·hr^{-1}
⊡**pentcia·ang·velocitel**	= ⊡$p\downarrow\measuredangle vc\ell$	= 1 rad·dwell^{-1}	= 12_{d} rad·day^{-1}
⊡**hexcia·ang·velocitel**	= ⊡$h\downarrow\measuredangle vc\ell$	= 1 rad·day^{-1}	

Angular Acceleration. The rotational analog of linear acceleration is *angular acceleration* (symbolized α), which is the time rate of change of angular velocity.

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{\check{r}}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{a_\perp}{\check{r}} \qquad a_\perp = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \qquad \text{(Primel)}$$

where $a_\perp$ is the tangential (linear) acceleration. Note that, because SI uses $\breve{\omega}$ rather than true ω, and radius r rather than radiality $\check{r}$, its version of angular acceleration is $\breve{\alpha}$ rather than true α:

$$\breve{\alpha} = \frac{\mathrm{d}\breve{\omega}}{\mathrm{d}t} = \frac{1}{r}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{a_\perp}{r} \qquad a_\perp = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \qquad \text{(SI)}$$

This gives $\breve{\alpha}$ a problematic dimensionality indistinguishable from that of frequency squared (inverse time squared).

Similar analysis applies when considering angular acceleration as a vector in three dimensions:

$$\boldsymbol{\alpha} = \frac{\check{\mathbf{r}} \times \mathbf{a}}{\|\check{\mathbf{r}}\|^2} \qquad \mathbf{a}_\perp = \boldsymbol{\alpha} \times \check{\mathbf{r}} \qquad \text{(Primel)}$$

$$\breve{\boldsymbol{\alpha}} = \frac{\mathbf{r} \times \mathbf{a}}{\|\mathbf{r}\|^2} \qquad \mathbf{a}_\perp = \breve{\boldsymbol{\alpha}} \times \mathbf{r} \qquad \text{(SI)}$$

Primel's coherent unit of angular acceleration is the ⊡**radielic·accelerel**, abbreviated ⊡*rdℓ\accℓ*:

$$⊡\textbf{radielic·accelerel} = \frac{\textbf{radian}}{⊡\textbf{lengthel}} \cdot \frac{⊡\textbf{lengthel}}{⊡\textbf{timel}^2} = \frac{\textbf{radian}}{⊡\textbf{timel}^2}$$

A reasonable synonym for this is ⊡**ang·accelerel**, abbreviated ⊡ $\measuredangle$*accℓ*. This is equivalent to 1194.3936_{d} rad·s^{-2}.

Angular Mass or Moment of Inertia. The rotational analog of mass is known as *moment of inertia* or *angular mass*. SI defines this as the mass m of the rotating object times the square of the radius r of rotation. But Primel uses the radiality $\check{r}$ instead:

$$I = mr^2 \quad \text{(SI)} \qquad \check{\check{I}} = m\check{r}^2 \quad \text{(Primel)}$$

Thus Primel's definition of angular mass is actually SI's version with the radianic operator applied *twice*.

Primel's coherent unit of angular mass is the ⊡**squaradiel·massel**, abbreviated ⊡*sdℓ·msℓ*:

$$\boxdot\textbf{squaradiel·massel} = \frac{\boxdot\textbf{lengthel}^2}{\textbf{radian}^2}\cdot\boxdot\textbf{massel}$$

A reasonable synonym for this is ⊡**ang·massel**, abbreviated ⊡ ∡*msℓ*. This is equivalent to approximately $0.371200648827305_{\mathrm{d}}$ $\mathrm{g{\cdot}cm^2rad^{-2}}$. A colloquial synonym for this unit is the ⊡**morsel·ang·mass**.

A useful scaling of this unit is the ⊡**pentqua·ang·massel**, which may be called colloquially the ⊡**hand·ang·mass**, because:

$$\begin{aligned}
\boxdot\textbf{hand·ang·mass} &= \boxdot\textbf{hand·squaradiality}\cdot\boxdot\textbf{hand·mass}\\
&= \boxdot\textbf{biqua·squaradiel}\cdot\boxdot\textbf{triqua·massel}\\
&= \boxdot\textbf{pentqua·squaradiel·massel}\\
&= \boxdot\textbf{pentqua·ang·massel}
\end{aligned}$$

This is equivalent to approximately $0.923665998489959_{\mathrm{d}}$ $\mathrm{kg{\cdot}dm^2rad^{-2}}$.

Angular Momentum. The rotational analog of momentum is known as *angular momentum*. SI symbolizes this as $\mathbf{L}$ and calculates it as the cross-product of the radius vector $\mathbf{r}$ (position vector of the rotating object relative to the axis of rotation) times the linear momentum vector $\mathbf{p}$. SI can also calculate it as the product of its version of angular mass I multiplied by its version of angular velocity vector $\breve{\boldsymbol{\omega}}$ (sans radians). This is by direct analogy with linear momentum being the product of mass m and linear velocity vector $\mathbf{v}$:

$$\mathbf{L} = I\breve{\boldsymbol{\omega}} = \mathbf{r}\times\mathbf{p} \qquad \mathbf{p} = m\mathbf{v} \qquad \text{(SI)}$$

However, this gives $\mathbf{L}$ a problematic dimensionality indistinguishable from action (which is linear momentum times linear displacement).

In contrast, Primel symbolizes its version of angular momentum as $\check{\mathbf{L}}$ and calculates it as the cross product of the *radiality* vector of the rotating object multiplied by its linear momentum vector $\mathbf{p}$. Primel can also calculate it as the product of its squaradianic version of angular mass $\check{\check{I}}$, times its true version of angular velocity vector $\boldsymbol{\omega}$ (with radians):

$$\check{\mathbf{L}} = \check{\check{I}}\boldsymbol{\omega} = \check{\mathbf{r}}\times\mathbf{p} \qquad \mathbf{p} = m\mathbf{v} \qquad \text{(Primel)}$$

This gives $\check{\mathbf{L}}$ its own unique dimensionality equivalent to action per angular displacement, analogous to linear momentum being equivalent to action per linear displacement.

This distinction between action and angular momentum is at the heart of why the Planck constant seems to have two values: h and $\hbar = h/\tau$. Both of these are expressed in units of action, but in reality physicists should be using $\check{\hbar} = \hbar/\mathrm{rad} = h/(\tau\ \mathrm{rad}) = h/\mathrm{turn}$. In other words, when we include the missing angular units, both h and $\hbar$ express the *same* constant: $\check{\hbar}$, the quantum of angular momentum.

Primel's coherent unit of angular momentum is the $\boxdot$**radiel·momel**,[10] abbreviated $\boxdot$*rdℓ·mmℓ*:

$$\boxdot\textbf{radiel·momel} = \frac{\boxdot\textbf{lengthel}}{\textbf{radian}} \cdot \frac{\boxdot\textbf{massel} \cdot \boxdot\textbf{lengthel}}{\boxdot\textbf{timel}}$$
$$= \frac{\boxdot\textbf{massel} \cdot \boxdot\textbf{lengthel}^2}{\boxdot\textbf{timel} \cdot \textbf{radian}} = \frac{\boxdot\textbf{actionel}}{\textbf{radian}}$$

A reasonable synonym for this is $\boxdot$**ang·momel**, abbreviated $\boxdot\measuredangle$*mmℓ*.

$$\begin{aligned}\boxdot\textbf{ang·momel} &= \boxdot\textbf{ang·massel} \cdot \boxdot\textbf{ang·velocitel}\\ &= \boxdot\textbf{squaradiel·massel} \cdot \boxdot\textbf{radielic·velocitel}\\ &= \boxdot\textbf{radiel·momel}\end{aligned}$$

This is equivalent to approximately $12.8286944234717_{\text{d}}$ g·cm^{2}s^{-1}rad^{-1}. A colloquial synonym for this unit is $\boxdot$**morsel·ang·momentum**.

A useful scaling of this unit is the $\boxdot$**quadqua·ang·momel**, which may be called colloquially the $\boxdot$**hand·ang·momentum**, because:

$$\begin{aligned}\boxdot\textbf{hand·ang·momentum} \quad &= \boxdot\textbf{hand·radiality} \cdot \boxdot\textbf{hand·momentum}\\ &= \boxdot\textbf{unqua·radiel} \cdot \boxdot\textbf{triqua·momel}\\ &= \boxdot\textbf{quadqua·radiel·momel}\\ &= \boxdot\textbf{quadqua·ang·momel}\end{aligned}$$

This is equivalent to approximately $2.66015807565108_{\text{d}}$ kg·m^{2}s^{-1}rad^{-1}.

Angular Force. The rotational analog of force is known as *angular force* or *torque*. SI symbolizes this as $\mathbf{T}$,[11] and calculates it as the cross-product of the radius vector $\mathbf{r}$ (position vector of the rotating object relative to the axis of rotation) times the linear force vector $\mathbf{F}$. SI can also calculate it as the product of its version of angular mass I multiplied by its version of angular acceleration vector $\breve{\boldsymbol{\alpha}}$ (sans radians). This is by direct analogy with linear force being the product of mass m and linear acceleration vector $\mathbf{a}$:

$$\mathbf{T} = I\breve{\boldsymbol{\alpha}} = \mathbf{r} \times \mathbf{F} \qquad \mathbf{F} = m\mathbf{a} \qquad \text{(SI)}$$

However, this gives $\mathbf{T}$ a problematic dimensionality indistinguishable from work (which is linear force times linear displacement).

In contrast, Primel symbolizes its version of angular force as $\check{\mathbf{T}}$ and calculates it as the cross product of the *radiality* vector of the rotating object multiplied by its linear force vector $\mathbf{F}$. Primel can also calculate it as the product of its squaradianic version of angular mass $\check{\check{I}}$, times its true version of angular acceleration vector $\boldsymbol{\alpha}$ (with radians):

$$\check{\mathbf{T}} = \check{\check{I}}\boldsymbol{\alpha} = \check{\mathbf{r}} \times \mathbf{F} \qquad \mathbf{F} = m\mathbf{a} \qquad \text{(Primel)}$$

[10] Recall that Primel's coherent unit of linear momentum, $\boxdot$**momentumel**, can be contracted to $\boxdot$**momel**.

[11] This is uppercase Greek tau, indistinguishable from Latin T. SI actually uses lowercase tau ($\boldsymbol{\tau}$) for this. But I am avoiding this in order to reserve $\tau = 2\pi$ as a circle constant.

This gives $\check{\mathbf{T}}$ its own unique dimensionality equivalent to work (i.e., energy) per angular displacement, analogous to linear force being equivalent to work per linear displacement.

Primel's coherent unit of angular force is the ⊡**radiel·forcel**, abbreviated ⊡$r d\ell \cdot f c\ell$:

$$\boxdot\textbf{radiel·forcel} = \frac{\boxdot\textbf{lengthel}}{\textbf{radian}} \cdot \frac{\boxdot\textbf{massel} \cdot \boxdot\textbf{lengthel}}{\boxdot\textbf{timel}^2}$$

$$= \frac{\boxdot\textbf{massel} \cdot \boxdot\textbf{lengthel}^2}{\boxdot\textbf{timel}^2 \cdot \textbf{radian}} = \frac{\boxdot\textbf{workel}}{\textbf{radian}}$$

A reasonable synonym for this is ⊡**ang·forcel**, abbreviated ⊡$\measuredangle f c\ell$.

$$\begin{aligned}\boxdot\textbf{ang·forcel} &= \boxdot\textbf{ang·massel} \cdot \boxdot\textbf{ang·accelerel}\\ &= \boxdot\textbf{squaradiel·massel} \cdot \boxdot\textbf{radielic·accelerel}\\ &= \boxdot\textbf{radiel·forcel}\end{aligned}$$

This is equivalent to approximately $443.359679275181_{\mathrm{d}}$ erg·rad^{-1} or $44.3359679275181_{\mathrm{d}}$ μJ·rad^{-1}.

One useful scaling is the ⊡**quadqua·radiel·forcel**, which can have the colloquial synonym ⊡**hand·ang·forcel**, because:

$$\begin{aligned}\boxdot\textbf{hand·ang·force} &= \boxdot\textbf{hand·radiality} \cdot \boxdot\textbf{hand·force}\\ &= \boxdot\textbf{unqua·radiel} \cdot \boxdot\textbf{triqua·forcel}\\ &= \boxdot\textbf{quadqua·radiel·forcel}\\ &= \boxdot\textbf{quadqua·ang·forcel}\end{aligned}$$

This is equivalent to approximately $0.919350630945015_{\mathrm{d}}$ J·rad^{-1}.

Translational and Rotational Kinetic Energy. Primel's versions of angular mass ($\check{\check{I}}$) and angular velocity (ω) can be used to calculate the rotational component of an object's kinetic energy (E_{R}). This is directly analogous to how the translational component of kinetic energy (E_{T}) can be calculated from the object's mass (m) and linear velocity (v):

$$E_{\mathrm{R}} = \frac{1}{2}\check{\check{I}}\omega^2 \qquad\qquad E_{\mathrm{T}} = \frac{1}{2}mv^2$$

We can confirm this by looking at Primel's units for these equalities:

$$\begin{aligned}\boxdot\textbf{energel} &= \boxdot\textbf{squaradiel·massel} \cdot \boxdot\textbf{radielic·velocitel}^2\\ &= \left(\boxdot\textbf{radiel}^2 \cdot \boxdot\textbf{massel}\right)\left(\boxdot\textbf{radiel}^{-2} \cdot \boxdot\textbf{velocitel}^2\right)\\ &= \boxdot\textbf{massel} \cdot \boxdot\textbf{velocitel}^2 = \boxdot\textbf{energel}\end{aligned}$$

Reusability of Unit Names. All of the quantitel and quantitelic unit names described above have been specific to the Primel metrology, so they have all sported Primel's brand mark ⊡. Nevertheless, they can be reused for other metrologies by simply

Primel ⊡ Summary of Angular Mechanical Units				
Quantity	**Quantitel**	**Abbrev**	**Decomposition**	**Metric Equivalents**
Plane Angle Angular Displacement	radian ⊡radielic·lengthel ⊡ang·lengthel	rad $\boxdot rd\ell\backslash lg\ell$ $\boxdot \measuredangle lg\ell$	rad	1 rad
Square Angle Solid Angle Angular Area	squaradian steradian ⊡squaradielic·areanel ⊡steradielic·areanel ⊡ang·areanel	sr $\boxdot sd\ell\backslash ar\ell$ $\boxdot \measuredangle ar\ell$	rad^2	1 rad^2 1 sr
Radiality	⊡radiel	$\boxdot rd\ell$	$\frac{\boxdot lg\ell}{rad}$	$8.20208\overline{3}_{\text{d}}\ \frac{\text{mm}}{\text{rad}}$
Inverse Radiality	⊡radielic	$\boxdot rd\ell\backslash$	$\frac{rad}{\boxdot lg\ell}$	$\approx 1.219202438404877_{\text{d}}\ \frac{\text{rad}}{\text{cm}}$
Squaradiality Steradiality	⊡squaradiel ⊡steradiel	$\boxdot sd\ell$	$\frac{\boxdot lg\ell^2}{rad^2}$	$67.2741710069\overline{4}_{\text{d}}\ \frac{\text{mm}^2}{\text{rad}^2}$
Inverse Squaradiality Inverse Steradiality	⊡squaradielic ⊡steradielic	$\boxdot sd\ell\backslash$	$\frac{rad^2}{\boxdot lg\ell^2}$	$\approx 1.4864545858124_{\text{d}}\ \frac{\text{rad}^2}{\text{cm}^2}$
Angular Velocity	⊡radielic·velocitel ⊡ang·velocitel	$\boxdot rd\ell\backslash vc\ell$ $\boxdot \measuredangle vc\ell$	$\frac{rad}{\boxdot tm\ell}$	$34.56_{\text{d}}\ \frac{\text{rad}}{\text{s}}$
Angular Acceleration	⊡radielic·accelerel ⊡ang·accelerel	$\boxdot rd\ell\backslash acc\ell$ $\boxdot \measuredangle acc\ell$	$\frac{rad}{\boxdot tm\ell^2}$	$1194.3936_{\text{d}}\ \frac{\text{rad}}{\text{s}^2}$
Angular Mass Moment of Inertia	⊡squaradiel·massel	$\boxdot sd\ell{\cdot}ms\ell$ $\boxdot \measuredangle ms\ell$	$\frac{\boxdot ms\ell{\cdot}\boxdot lg\ell^2}{rad^2}$	$\approx 0.371200648827305_{\text{d}}\ \frac{\text{g}{\cdot}\text{cm}^2}{\text{rad}^2}$
Angular Momentum	⊡radiel·momel ⊡ang·momel	$\boxdot rd\ell{\cdot}mm\ell$ $\boxdot \measuredangle mm\ell$	$\frac{\boxdot ms\ell{\cdot}\boxdot lg\ell^2}{\boxdot tm\ell{\cdot}rad}$	$\approx 12.8286944234717_{\text{d}}\ \frac{\text{g}{\cdot}\text{cm}^2}{\text{s}{\cdot}\text{rad}}$
Angular Force Torque	⊡radiel·forcel ⊡ang·forcel	$\boxdot rd\ell{\cdot}fc\ell$ $\boxdot \measuredangle fc\ell$	$\frac{\boxdot ms\ell{\cdot}\boxdot lg\ell^2}{\boxdot tm\ell^2{\cdot}rad}$	$\approx 44.3359679275181_{\text{d}}\ \frac{\mu\text{J}}{\text{rad}}$

replacing Primel's brand mark with the brand mark of another metrology (and of course recomputing the unit sizes in terms of that metrology's own "mundane realities").

Angular Transcendental Functions

An additional problem encountered when considering the dimensionality of angular displacement is the question of how to treat trigonometric and other functions that take angular displacements as arguments. Such functions often have Taylor series expansions, which are infinite sums of terms each containing differing powers of the argument. Such terms would not be commensurate with each other, and so could not be added together, unless the original argument were dimensionless. To see the solution to this issue, let us first consider how to apply the radianic operator to a mathematical function.

For any unary function f, let us define its radianic $\check{f}$ as equivalent to the original function applied to the radianic of its argument:

$$\check{f}(\theta) = f\left(\check{\theta}\right) = f\left(\frac{\theta}{1\ \text{rad}}\right)$$

In this way, we have introduced a new "complete" function $\breve{f}$ which can accept a dimensioned angular quantity θ. But the first step of $\breve{f}$ is to extract the dimensionless radianic measure quantity $\breve{\theta}$ and pass that on to f. Then f itself acts as an auxiliary function which only accepts a dimensionless argument, but which is thereby free to calculate any terms it likes from it. If the function is transcendental then it is free to do a Taylor series expansion:

$$\breve{\sin}\,\theta = \sin\breve{\theta} = \sum_{n=0}^{\infty}(-1)^n\frac{\breve{\theta}^{2n+1}}{(2n+1)!} = \breve{\theta} - \frac{\breve{\theta}^3}{3!} + \frac{\breve{\theta}^5}{5!} - \frac{\breve{\theta}^7}{7!} + \cdots$$

$$\breve{\cos}\,\theta = \cos\breve{\theta} = \sum_{n=0}^{\infty}(-1)^n\frac{\breve{\theta}^{2n}}{(2n)!} = 1 - \frac{\breve{\theta}^2}{2!} + \frac{\breve{\theta}^4}{4!} - \frac{\breve{\theta}^6}{6!} + \cdots$$

$$\breve{\operatorname{expi}}\,\theta = \operatorname{expi}\breve{\theta} = e^{i\breve{\theta}} = \sum_{n=0}^{\infty} i^n\frac{\breve{\theta}^n}{n!} = 1 + i\breve{\theta} - \frac{\breve{\theta}^2}{2!} - i\frac{\breve{\theta}^3}{3!} + \frac{\breve{\theta}^4}{4!} + i\frac{\breve{\theta}^5}{5!} - \cdots$$

$$= \left(1 - \frac{\breve{\theta}^2}{2!} + \frac{\breve{\theta}^4}{4!} - \cdots\right) + i\left(\breve{\theta} - \frac{\breve{\theta}^3}{3!} + \frac{\breve{\theta}^5}{5!} - \cdots\right) = \cos\breve{\theta} + i\sin\breve{\theta} = \breve{\cos}\,\theta + i\,\breve{\sin}\,\theta$$

The heart of the matter is that transcendental functions like sin, cos and expi actually compute relationships between pure mathematical abstractions with no regard to any sort of physical manifestation in the real world. Not even the geometric interpretation as angles in the plane is really pertinent to the abstraction (although it can provide intuition to aid in understanding it).

On the other hand, functions such as $\breve{\sin}$, $\breve{\cos}$, and $\breve{\operatorname{expi}}$ do relate to real physical quantities that can actually be measured. But to further analyze these quantities, we need to strip them of their physical dimensionality and bridge into the world of pure mathematics. This is fine, but it does mean that we have identified two quite different, yet related, kinds of function, applicable to very different contexts. Problems only arise when we try to conflate the two.

More to Come

Designing Primel's angular mechanics units has required diverging from SI's approach to the subject, but as it turns out, not in ways that were entirely unprecedented. The next issue will cover Primel's units for electricity and magnetism. I will examine not only the usual types of quantities encountered by first-year physics and engineering students, but in fact all the various kinds of quantities embodied in famous equations by James Clerk Maxwell and others.[12] Primel's approach does not in any way change the dimensionality of these quantities. However, in order to derive a balanced system of quantitel unit names for them, it does make interesting changes to the *terminology* applied to electromagnetic quantities. This may actually prove to be the most controversial aspect of the metrology. I hope you will all get a *charge* out of it.

[12] See https://en.wikipedia.org/wiki/Maxwell's_equations.

[z] • SYSTEMATIC DOZENAL NOMENCLATURE SUMMARY • [z]

N	ROOT	ABBR	MULTIPLIER PREFIX $N\times$	RECIPROCAL PREFIX $\frac{1}{N}\times$	POWER PREFIXES POSITIVE $10^{+N}\times$	POWER PREFIXES NEGATIVE $10^{-N}\times$
0	nil	n	nili·	nilinfra·	nilqua·	nilcia·
1	un	u	uni·	uninfra·	unqua·	uncia·
2	bi	b	bina·	bininfra·	biqua·	bicia·
3	tri	t	trina·	trininfra·	triqua·	tricia·
4	quad	q	quadra·	quadinfra·	quadqua·	quadcia·
5	pent	p	penta·	pentinfra·	pentqua·	pentcia·
6	hex	h	hexa·	hexinfra·	hexqua·	hexcia·
7	sept	s	septa·	septinfra·	septqua·	septcia·
8	oct	o	octa·	octinfra·	octqua·	octcia·
9	enn	e	ennea·	enninfra·	ennqua·	enncia·
↊	dec	d	deca·	decinfra·	decqua·	deccia·
↋	lev	L	leva·	levinfra·	levqua·	levcia·
10	unnil	un	unnili·	unnilinfra·	unnilqua·	unnilcia·
11	unun	uu	ununi·	ununinfra·	ununqua·	ununcia·
12	unbi	ub	unbina·	unbininfra·	unbiqua·	unbicia·
13	untri	ut	untrina·	untrininfra·	untriqua·	untricia·
14	unquad	uq	unquadra·	unquadinfra·	unquadqua·	unquadcia·
15	unpent	up	unpenta·	unpentinfra·	unpentqua·	unpentcia·
16	unhex	uh	unhexa·	unhexinfra·	unhexqua·	unhexcia·
17	unsept	us	unsepta·	unseptinfra·	unseptqua·	unseptcia·
18	unoct	uo	unocta·	unoctinfra·	unoctqua·	unoctcia·
19	unenn	ue	unennea·	unenninfra·	unennqua·	unenncia·
1↊	undec	ud	undeca·	undecinfra·	undecqua·	undeccia·
1↋	unlev	uL	unleva·	unlevinfra·	unlevqua·	unlevcia·
20	binil	bn	binili·	binilinfra·	binilqua·	binilcia·

etc...

uncia was Latin for *one twelfth* • retains same meaning • *inch* and *ounce* are English derivatives
Concatenating roots = positional place-value • Suggested pronunciation: -cia = /ʃə/ ("-sha")
Concatenating prefixes = multiplication • mix & match freely • Commutative Law applies
Prefer Unicode abbreviations where supported • ASCII abbreviations for email, text, etc.

SDN FORM	EXAMPLE VALUE [z]	EXAMPLE SDN	ABBREVIATION UNICODE	ABBREVIATION ASCII
Root Form	46	quadhex	qh	`qh`
Multiplier Prefix	$46\times$	quadhexa·	qh•	`qh*`
With Fractional Part	$4.6\times$	quad.dot.hexa·	q.h•	`q.h*`
Ordinal	46^{th}	quadhexal	qh′	`qh'`
Reciprocal Prefix	$\frac{1}{46}\times$	quadhexinfra·	qh\	`qh\`
Positive Power Prefix	$10^{+46}\times$	quadhexqua·	qh↑	`qh@`
Negative Power Prefix	$10^{-46}\times$	quadhexcia·	qh↓	`qh#`
Rational Number	$4\times\frac{1}{5}\times$	quadra·pentinfra·	q•p\	`q*p\`
Rational Number	$\frac{1}{5}\times4\times$	pentinfra·quadra·	p\q•	`p\q*`
Scientific Notation	$4\times10^{+6}\times$	quadra·hexqua·	q•h↑	`q*h@`
With Fractional Part	$4.5\times10^{+6}\times$	quad.dot.penta·hexqua·	q.p•h↑	`q.p*h@`
Scientific Notation	$10^{+6}\times4\times$	hexqua·quadra·	h↑q•	`h@q*`
With Fractional Part	$10^{+6}\times4.5\times$	hexqua·quad.dot.penta·	h↑q.p•	`h@q.p*`
one dozen years	$10^{+1}\times$ year	unqua·year, unquennium	u↑yr	`u@yr`
one gross years	$10^{+2}\times$ year	biqua·year, biquennium	b↑yr	`b@yr`
one galore years	$10^{+3}\times$ year	triqua·year, triquennium	t↑yr	`t@yr`
two hours (a "dwell")	$10^{-1}\times$ day	uncia·day	u↓dy	`u#dy`
ten minutes (a "breather")	$10^{-2}\times$ day	bicia·day	b↓dy	`b#dy`
fifty seconds (a "trice")	$10^{-3}\times$ day	tricia·day	t↓dy	`t#dy`

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Original thread: https://www.tapatalk.com/groups/dozensonline/systematic-dozenal-nomenclature-t463.html

An Unpublished Manuscript of Leibniz's on Duodecimal

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In the field of mathematics, the German polymath Gottfried Wilhelm Leibniz (1646–1716)[1] is best known for his independent inventions of the calculus and binary arithmetic. Less well known is that Leibniz also invented the base-16 number system, which he called "sedecimal" (see Strickland and Jones 2022). During his exploration of bases 2 and 16, he had occasion to work a little on other number bases, including the duodecimal. While most of his references to base 12 occur in writings that deal with other topics as well, there does exist one manuscript exclusively devoted to it. An English translation of that manuscript is given at the end of this article, the first time the manuscript has been published in any language. By way of an introduction, and to provide some context, I shall begin with an outline of Leibniz's other engagements with duodecimal, many of which are also unpublished. In contrast to his writings on the binary system, which span hundreds of manuscript pages, Leibniz's treatment of duodecimal is occasional, scattered, and unsystematic. Yet from those it is clear that Leibniz had a sound understanding of duodecimal, and as we shall see, duodecimal may even have had a role in his invention of binary.

The earliest mention of duodecimal in Leibniz's writings occurs in a preface he wrote in 1670 to an edition of the writings of the Italian humanist Marius Nizolius (1498–1576). There he considers the view of those who would have it that truth depends upon the definitions of terms, and definitions of terms in turn upon the human mind (in other words, that truth is arbitrary). To this Leibniz (1969, 128) responds: "In arithmetic, and in other disciplines as well, truths remain the same even if notations are changed, and it does not matter whether a decimal or a duodecimal number system is used." Unfortunately, Leibniz does not reveal where he had learned about duodecimal, though his use of it suggests he thought it was sufficiently well known that educated readers would be able to understand his argument.

Between 1672 and 1676 Leibniz lived in Paris, studying under the tutelage of some of the foremost mathematicians of the day, most notably Christiaan Huygens (1629–1695). As a result of his intense studies he devised the calculus in 1675, and binary arithmetic a few years later. The first references to duodecimal in his mathematical writings date from around this time. For example, in a manuscript entitled "Thesaurus mathematicus" [Mathematical Thesaurus], probably written in either 1678 or 1679, Leibniz works through various topics in arithmetic, geometry, and mechanics; near the end, he outlines how positional notation works in the decimal and then the duodecimal number system:

> From this outline it is clear that only these ten digits are needed: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Those in the first position signify the equivalent number of 1s, namely no 1s, one 1, two 1s, three, four, five, six, seven, eight, nine 1s. Those in the second position signify the equivalent number of *10*s, that is, 1s taken ten times; in the third position, the equivalent number of *100*s, that is, *10*s taken ten times,

[1] *EDITOR'S NOTE:* All numerals in this article default to decimal [d], unless the text explicitly describes use of duodecimal by Leibniz or his contemporaries. Symbols shown for transdecimal digits are those used by the historical figures themselves.

> or the squares of 10; in the fourth position, the equivalent number of *1000*s, that is, *100*s taken ten times, or the cubes of 10, and so on. And in place of 10 one would be able to put any other number, for example, 12. For just as when the base a is 10, the square a^2 signifies 100 and the cube a^3 signifies 1000, so when a is 12, a^2 will be 12 times 12, that is, 144, and a^3 will be 12 times 144. But on this method, instead of the digits mentioned above—0, 1 etc. 9—two new digits would be needed in addition, one which would represent ten, the other which would represent eleven; but [the digits] 10 would signify twelve, and 100 would signify one hundred and forty four. And there are some who prefer to use this method of calculating over the common method, because 12 can be divided by 2, 3, 4, and 6; in addition, a calculation is completed with fewer digits. But the difference is not so great as to be worth abandoning the decimal progression. (LH 35, 1, 25 Bl. 3v)

Leibniz often repeated the claim that some people had preferred duodecimal to decimal, but it is not clear who he had in mind. Late in life he claimed that a proponent of duodecimal had been identified by the German mathematician Daniel Schwenter (1585–1636): "In the German *Deliciae mathematicae*, someone is reported to have given preference to the duodecimal progression, in which eleven digits will be needed, namely 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, δ, ε, where δ is 10 and ε is 11" (LBr 705 Bl. 93r). The reference is probably to Schwenter's *Deliciae physico-mathematicae* [*The Charms of Physico-Mathematics*] of 1636, which was posthumously revised and expanded by Georg Philipp Harsdörffer (1607–1658) in 1651 and again in 1653. However, as far as I have been able to tell, in none of those works is there any mention of the duodecimal system, let alone any report of anyone endorsing it.[2] Alternatively, when referring to proponents of duodecimal, Leibniz may have been thinking of Blaise Pascal (1623–1662), who had mentioned duodecimal as an alternative to decimal in an essay presented to the Academie Parisienne in 1654 and posthumously published eleven years later. In that essay, Pascal (1665, 42) promised to give a method to determine whether a given number is divisible by any other number, insisting that it would work "not just in our decimal system of numeration (which has been established not as a result of natural necessity, as the common man thinks, but as a result of human custom, and quite foolishly, to be sure), but in a system of numeration based on any progression whatsoever." To illustrate, he applied his method to the duodecimal system. Leibniz was certainly aware of Pascal's essay, and may have taken Pascal's work with duodecimal as an endorsement thereof. He certainly appears to have absorbed what Pascal had to say about duodecimal; in a manuscript written around 1678, Leibniz noted that if the duodecimal system were used, the arithmetic checking method known as casting out nines "could become the proof by casting out elevens" (LH 35, 4, 13 Bl. 21), apparently borrowing the observation from Pascal (1665, 47–48), who had made it some years before.

In the spring of 1680, Leibniz met the Dutch mathematician Johann Jakob Ferguson (1630–1706), and must have mentioned both binary and duodecimal to him, as in the scratch paper upon which both recorded their ideas, Leibniz (1976, 137) wrote out a table showing the values of the decimal numbers 0–8 in binary, and a set of duodecimal digits, with the two extra digits given as ~ and $. In August of the same year, Leibniz noted: "It is well known that all fractions can be expressed by an infinite sequence of integers of a certain progression, for example, the decimal, or even the duodecimal,

[2] Schwenter (1636, 117–122) does, however, provide a lengthy list of ways that the numbers 2–30 are embedded or reflected in the physical and spiritual realms. With 12, for example, he notes that there were 12 tribes of Israel and 12 apostles, that there are 12 signs of the zodiac, that the year is divided into 12 months etc.

or the one I prefer overall, the binary" (LH 35, 13, 3 Bl. 33). The remainder of this manuscript is concerned with binary fractions, but given the confidence of his remark here, it is likely that he had already undertaken some investigation of duodecimal fractions. If he had, his work on that remains to be discovered.

If Leibniz himself is to be believed, duodecimal even played a role in his invention of binary. The story Leibniz would tell in the 1690s onwards was that he had hit upon binary as the simplest number system from conscious reflection upon the duodecimal and quaternary number systems. In 1697 he wrote:

> It is apparent that some considered the duodecimal to be more useful while others took pleasure in the Pythagorean tetractys. At some point it occurred to me to consider what would ultimately be revealed if we used the simplest of all [progressions], namely the dyadic or binary. (Strickland and Lewis 2022, 110)

(The "Pythagorean tetractys," by the way, is the quaternary—base 4—number system developed by Erhard Weigel (1673).) We should be cautious of at least some of what Leibniz claims here. Certainly, there is no evidence that he knew of Weigel's quaternary system before 1683, several years after he had invented binary, in which case he could not have been influenced by quaternary. But as we have seen, Leibniz did know about duodecimal at least as far back as 1670, and while the manuscript evidence he left behind does not enable us to verify his later claim that he found his way to binary via duodecimal, it does not enable us to rule it out either.

Although Leibniz wrote relatively little about duodecimal, he was clearly aware of its advantages over decimal, and occasionally indicated that, were the decimal system to be dislodged from common usage, it should be replaced either by the duodecimal or sedecimal. In 1694 or 1695 he wrote:

> I think that if anything were to be changed in practice, it would be to use the duodecimal or sedecimal instead of the decimal, for the larger the numbers used by a progression, the more convenient the calculation (Strickland and Lewis 2022, 85).

Leibniz later made the same claim in "Explanation of binary arithmetic," the only one of his many writings on binary published in his own lifetime. There, after outlining binary notation and arithmetic, Leibniz insisted that binary was not intended to replace decimal in everyday usage because the long strings of digits made it impractical, in which case, he said, it is better to stick with decimal because the numbers are not as long. He then stated: "And if we were accustomed to proceed by twelves or sixteens, there would be even more benefit" (Strickland and Lewis 2022, 196). Despite publicly acknowledging the advantages of duodecimal and sedecimal, Leibniz was no vocal advocate of wholesale reform, nor did he make much use of these number systems in all but a small handful of his extensive mathematical writings.

Let us turn, then, to Leibniz's unpublished manuscript on duodecimal. It begins by noting that, in the decimal system, the digital root (that is, the digit sum) of multiples of nine is always nine, e.g. $9 \times 3 = 27$, and $2 + 7 = 9$. Leibniz then generalizes this to any number base, or "progression," supposing that for any base n, the digital root of multiples of $n - 1$ is always $n - 1$. He illustrates this using the duodecimal (or "duodenary") system, showing that the digital root of any multiple of 11 is always 11, or rather, since he uses the Greek letters χ and ϕ for 10 and 11 respectively, the digital root for any multiple of ϕ is always ϕ. To secure the point, Leibniz draws a table of duodecimal numbers in which his "new notation" for duodecimal is shown alongside the "old meaning" (i.e. the decimal equivalents).

Unfortunately, Leibniz's motivation for writing the manuscript is unknown—he gives the impression of simply wanting to record an observation he had made, but does not reveal what inspired him to make the observation in the first place. When did he write the piece? The watermark of the manuscript is found in only two of his other writings, one thought to have been written in 1693, the other positively dated to June 1706, which suggests it was written sometime between those two dates. However, it was filed among Leibniz's mathematical papers of 1695, making it reasonable to think it was written around that time.[3] [4]

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[3] Work on this paper was graciously supported by the Gerda Henkel Stiftung (award AZ 46/V/21).

[4] *EDITOR'S NOTE:* The following pages include a transcription of the original Latin of Leibniz's manuscript, an English translation on the facing page, and finally an image of the original manuscript itself. The Dozenal Society of America would like to thank Professor Strickland for graciously choosing *The Duodecimal Bulletin* for first publication of this intriguing historical document.

Leibniz: De Progressione Duodenaria (c. 1695)[5]

Novenarii proprietas est, quae facit ut summa notarum in multiplis ejus novenarium rursus componat:

9 per	1	2	3	4	5	6	7	8	9	10
dat	9	18	27	36	45	54	63	72	81	90

Sed sciendum est hanc proprietatem oriri ex nostro notandi modo qui est arbitrarius; nempe ex eo quod progressione denaria utimur, et post novem redimus ad 1 adjecta 0. Potuissemus vero alia progressione quacunque uti et semper haec futura esset proprietas numeri ultimi in assumta progressione. Exempli causa, si pro denaria progressione adhiberetur duodenaria, ut a quibusdam curiositatis causa factum est, numerus undenarius simili proprietate gauderet, quod ostendere opera pretium erit. Nempe si duodenaria progressio adhiberetur, numeri decem et undecim proprias acciperent notas, veluti χ pro denario et ϕ pro undenario si placet. Itaque numeri usque ad duodecies duodecim seu centium quadraginta quatuor, ita stabunt:

Notatio novem:	1	2	3	4	5	6	7	8	9	χ	ϕ	10
Significatio antique:	1	2	3	4	5	6	7	8	9	10	11	12
Notatio novem:	11	12	13	14	15	16	17	18	19	1χ	1ϕ	20
Significatio antique:	13	14	15	16	17	18	19	20	21	22	23	24
Notatio novem:	21	22	23	24	25	26	27	28	29	2χ	2ϕ	30
Significatio antique:	25	26	27	28	29	30	31	32	33	34	35	36
Notatio novem:	31	32	33	34	35	36	37	38	39	3χ	3ϕ	40
Significatio antique:	37	38	39	40	41	42	43	44	45	46	47	48
Notatio novem:	41	42	43	44	45	46	47	48	49	4χ	4ϕ	50
Significatio antique:	49	50	51	52	53	54	55	56	57	58	49	60
Notatio novem:	51	52	53	54	55	56	57	58	59	5χ	5ϕ	60
Significatio antique:	61	62	63	64	65	66	67	68	69	70	71	72
Notatio novem:	61	62	63	64	65	66	67	68	69	6χ	6ϕ	70
Significatio antique:	73	74	75	76	77	78	79	80	81	82	83	84
Notatio novem:	71	72	73	74	75	76	77	78	79	7χ	7ϕ	80
Significatio antique:	85	86	87	88	89	90	91	92	93	94	95	96
Notatio novem:	81	82	83	84	85	86	87	88	89	8χ	8ϕ	90
Significatio antique:	97	98	99	100	101	102	103	104	105	106	107	108
Notatio novem:	91	92	93	94	95	96	97	98	99	9χ	9ϕ	$\chi0$
Significatio antique:	109	110	111	112	113	114	115	116	117	118	119	120
Notatio novem:	$\chi1$	$\chi2$	$\chi3$	$\chi4$	$\chi5$	$\chi6$	$\chi7$	$\chi8$	$\chi9$	$\chi\chi$	$\chi\phi$	$\phi0$
Significatio antique:	121	122	123	124	125	126	127	128	129	130	131	132
Notatio novem:	$\phi1$	$\phi2$	$\phi3$	$\phi4$	$\phi5$	$\phi6$	$\phi7$	$\phi8$	$\phi9$	$\phi\chi$	$\phi\phi$	100
Significatio antique:	133	134	135	136	137	138	139	140	141	142	143	144

Hinc jam multipli ipsius undenarii:

id est, communa notatione:	11	22	33	44	55	66	77	88	99	110	121	132
nova notatione erunt:	ϕ	1χ	29	38	47	56	65	74	83	92	$\chi1$	$\phi0$

ubi etiam summa binarum notarum semper facit undecim.

[5] LH 35, 12 1 Bl. 40r. The original Latin.

Leibniz: On the Duodenary Progression (c. 1695)[6]

It is a property of nines that the sum of the digits in its multiples makes nine again:

9 by	1	2	3	4	5	6	7	8	9	10
gives	9	18	27	36	45	54	63	72	81	90

But it should be known that this property originates from our way of writing, which is arbitrary, namely, from the fact that we use the decimal progression and after nine we return to 1 by adding 0. But we could use any other progression and the aforementioned property would always be the property of the last digit in the progression adopted. For example, if the duodenary progression were used in place of the decimal, as has been done by some people for the sake of curiosity, the number eleven would enjoy a similar property, which it will be worthwhile to show. Of course, if the duodenary progression were to be used, the numbers ten and eleven would have their own digits, such as χ for ten and ϕ for eleven, if you like. Therefore the numbers up to twelve times twelve, that is, one hundred and forty-four, will be as follows:

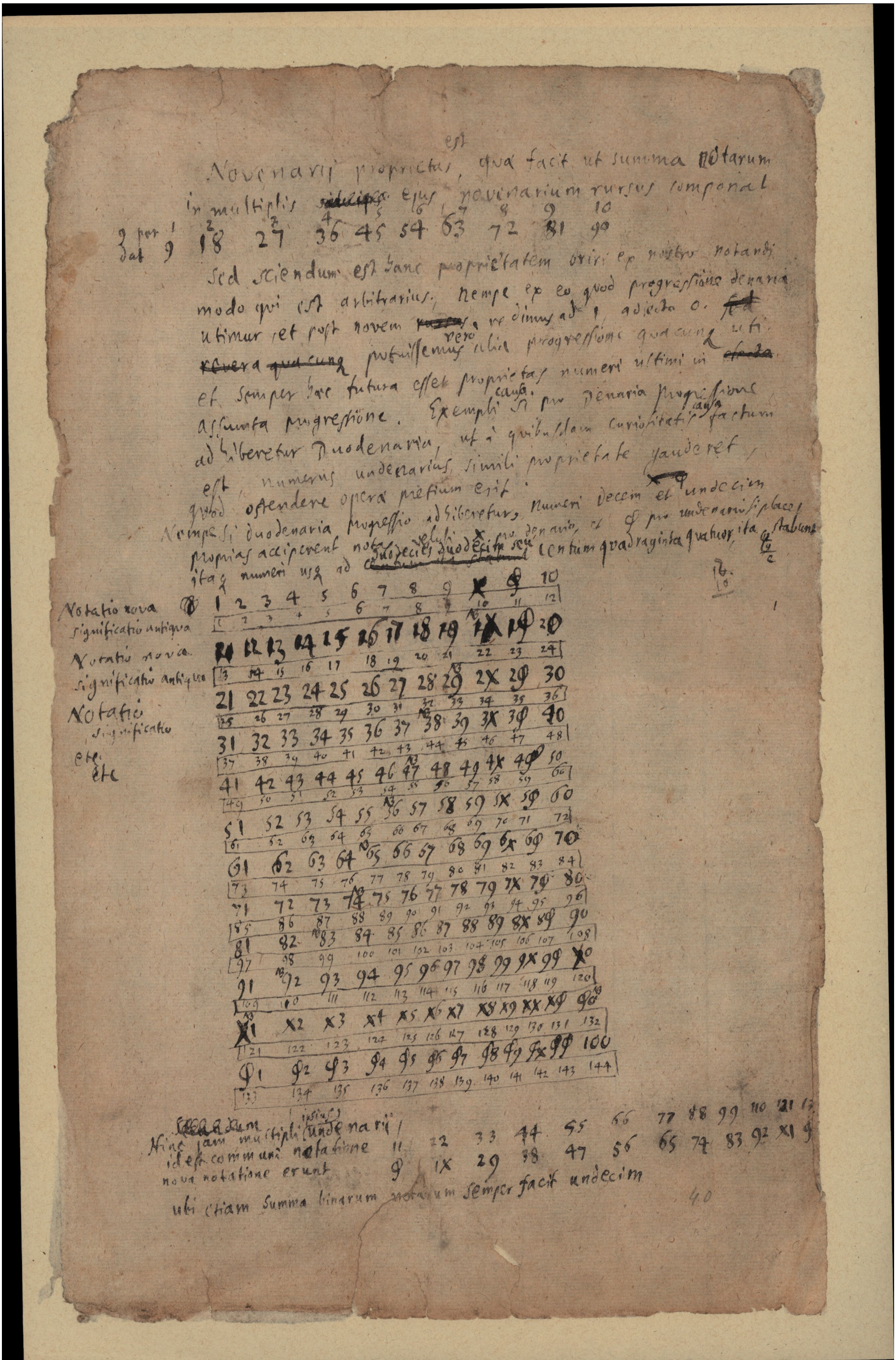
New notation:	1	2	3	4	5	6	7	8	9	χ	ϕ	10
Old meaning:	1	2	3	4	5	6	7	8	9	10	11	12
New notation:	11	12	13	14	15	16	17	18	19	1χ	1ϕ	20
Old meaning:	13	14	15	16	17	18	19	20	21	22	23	24
New notation:	21	22	23	24	25	26	27	28	29	2χ	2ϕ	30
Old meaning:	25	26	27	28	29	30	31	32	33	34	35	36
New notation:	31	32	33	34	35	36	37	38	39	3χ	3ϕ	40
Old meaning:	37	38	39	40	41	42	43	44	45	46	47	48
New notation:	41	42	43	44	45	46	47	48	49	4χ	4ϕ	50
Old meaning:	49	50	51	52	53	54	55	56	57	58	49	60
New notation:	51	52	53	54	55	56	57	58	59	5χ	5ϕ	60
Old meaning:	61	62	63	64	65	66	67	68	69	70	71	72
New notation:	61	62	63	64	65	66	67	68	69	6χ	6ϕ	70
Old meaning:	73	74	75	76	77	78	79	80	81	82	83	84
New notation:	71	72	73	74	75	76	77	78	79	7χ	7ϕ	80
Old meaning:	85	86	87	88	89	90	91	92	93	94	95	96
New notation:	81	82	83	84	85	86	87	88	89	8χ	8ϕ	90
Old meaning:	97	98	99	100	101	102	103	104	105	106	107	108
New notation:	91	92	93	94	95	96	97	98	99	9χ	9ϕ	$\chi0$
Old meaning:	109	110	111	112	113	114	115	116	117	118	119	120
New notation:	$\chi1$	$\chi2$	$\chi3$	$\chi4$	$\chi5$	$\chi6$	$\chi7$	$\chi8$	$\chi9$	$\chi\chi$	$\chi\phi$	$\phi0$
Old meaning:	121	122	123	124	125	126	127	128	129	130	131	132
New notation:	$\phi1$	$\phi2$	$\phi3$	$\phi4$	$\phi5$	$\phi6$	$\phi7$	$\phi8$	$\phi9$	$\phi\chi$	$\phi\phi$	100
Old meaning:	133	134	135	136	137	138	139	140	141	142	143	144

Hence now the multiples of eleven:

that is, in the common notation:	11	22	33	44	55	66	77	88	99	110	121	132
will be in the new notation:	ϕ	1χ	29	38	47	56	65	74	83	92	$\chi1$	$\phi0$

where also the sum of the two digits always makes eleven.

[6] LH 35, 12 1 Bl. 40r. Translated from the Latin.



Novenarii proprietas est, quae facit ut summa notarum in multiplis ejus, novenarium rursus componat

	1	2	3	4	5	6	7	8	9	10
9 per dat	9	18	27	36	45	54	63	72	81	90

Sed sciendum est hanc proprietatem oriri ex nostro notandi modo qui est arbitrarius, nempe ex eo quod progressione denaria utimur et post novem redimus ad 1, adjecto 0. Si vero potuissemus alia progressione quacunque uti et semper haec futura esset proprietas numeri ultimi in assumta progressione. Exempli causa si pro denaria progressione adhiberetur duodenaria, ut a quibusdam curiositatis causa factum est, numerus undenarius simili proprietate gauderet, quod ostendere operae pretium erit.

Nempe si duodenaria progressio adhiberetur, numeri decem et undecim propias acciperent notas, velut X pro denario, et ↋ pro undenario, itaque numeri usque ad centum quadraginta quatuor, ita stabunt

Notatio nova	1	2	3	4	5	6	7	8	9	X	↋	10
significatio antiqua	1	2	3	4	5	6	7	8	9	10	11	12
Notatio nova	11	12	13	14	15	16	17	18	19	1X	1↋	20
significatio antiqua	13	14	15	16	17	18	19	20	21	22	23	24
Notatio	21	22	23	24	25	26	27	28	29	2X	2↋	30
significatio	25	26	27	28	29	30	31	32	33	34	35	36
etc.	31	32	33	34	35	36	37	38	39	3X	3↋	40
etc.	37	38	39	40	41	42	43	44	45	46	47	48
	41	42	43	44	45	46	47	48	49	4X	4↋	50
	49	50	51	52	53	54	55	56	57	58	59	60
	51	52	53	54	55	56	57	58	59	5X	5↋	60
	61	62	63	64	65	66	67	68	69	70	71	72
	61	62	63	64	65	66	67	68	69	6X	6↋	70
	73	74	75	76	77	78	79	80	81	82	83	84
	71	72	73	74	75	76	77	78	79	7X	7↋	80
	85	86	87	88	89	90	91	92	93	94	95	96
	81	82	83	84	85	86	87	88	89	8X	8↋	90
	97	98	99	100	101	102	103	104	105	106	107	108
	91	92	93	94	95	96	97	98	99	9X	9↋	X0
	109	110	111	112	113	114	115	116	117	118	119	120
	X1	X2	X3	X4	X5	X6	X7	X8	X9	XX	X↋	↋0
	121	122	123	124	125	126	127	128	129	130	131	132
	↋1	↋2	↋3	↋4	↋5	↋6	↋7	↋8	↋9	↋X	↋↋	100
	133	134	135	136	137	138	139	140	141	142	143	144

Hinc jam multipli ipsius undenarii, id est communi notatione	11	22	33	44	55	66	77	88	99	110	121	132
nova notatione erunt	↋	1X	29	38	47	56	65	74	83	92	X1	↋0

ubi etiam summa binarum notarum semper facit undecim

40

Dozenal Magic Squares

by Jay L. Schiffman

Introductory Discussion

A magic square is a configuration such that the sum of the elements in every row, every column and along both diagonals is the same. This common sum is referred to as the *magic sum*. For example, if one places each of the first nine counting integers in the following array, a magic square is obtained:

4	3	8
9	5	1
2	7	6

A natural question one might ask is how the integers were placed in their respective cells. Observe that:

$$\sum_{i=1}^{9} i = 1+2+3+4+5+6+7+8+9 = 45_{\mathrm{d}} = 39_{\mathrm{z}}$$

using Gauss' method (or directly), and $45_{\mathrm{d}} = 39_{\mathrm{z}}$ must be apportioned equally among the three rows, which implied that the sum of the entries in each row, column, and diagonal is $\frac{45}{3}_{\mathrm{d}} = \frac{39}{3}_{\mathrm{z}} = 15_{\mathrm{d}} = 13_{\mathrm{z}}$.[1]

Our initial goal hence is to determine which number can be placed in the center square which encompasses four sums; for the entry in the $(2,2)$ position must occur in four different sums that total $15_{\mathrm{d}} = 13_{\mathrm{z}}$; namely from the second row, second column and both diagonals. The only candidate is 5. Observe that

$$15_{\mathrm{d}} = 13_{\mathrm{z}} = 1+5+9 = 2+5+8 = 3+5+7 = 4+5+6$$

We next focus on the entries situated on the diagonal corners. Such entries occur in three different sums that total $15_{\mathrm{d}} = 13_{\mathrm{z}}$. The entry in the (1,1) position must occur in sums from the first row, first column and the main diagonal. Similarly the entry in the (3,3) position occurs in sums from the third row, third column and the main diagonal. Likewise, the entry in the (1,3) position occurs in sums from the first row, third column and the off diagonal while the entry in the (3,1) position occurs in sums from the third row, first column and the off diagonal. The only possibilities for numbers in the diagonal corners are 2, 4, 6 and 8. Observe that

[1] For those readers who are unfamiliar with Gauss' Method, he wrote the sum of the first nine consecutive integer in the first row in the natural order. In the second row, he wrote the sum in reverse order. Adding the two rows, he obtained twice the sum he desired. One would notice that down every column, a sum of ten is obtained, and we have nine addends of ten for a sum of ninety (seven dozen six). Hence the desired sum is half of ninety (seven dozen six) or forty-five (three dozen nine).

$$15_d = 13_z = 2+4+9 = 2+5+8 = 2+6+7$$
$$15_d = 13_z = 4+2+9 = 4+3+8 = 4+5+6$$
$$15_d = 13_z = 6+1+8 = 6+2+7 = 6+4+5$$
$$15_d = 13_z = 8+1+6 = 8+2+5 = 8+3+4$$

Thus if 4 is placed in the (1,1) position, then 6 must be placed in the (3,3) position. Meanwhile if 8 is placed in the (1,3) position, then 2 must be placed in the (3,1) position. This leaves the integers 1, 3, 7 and 9 to be placed in the other cells that are not in the center cell or in the cells on the diagonal corners. These integers are addends in only two different sums. The entry in the (1,2) position occurs in sums from the first row and second column. The entry in the (2,1) position occurs in sums from the second row and first column. The entry in the (2,3) position occurs in sums from the second row and third column. Finally the entry in the (3,2) position occurs in sums from the third row and second column. We note that

$$15_d = 13_z = 1+5+6 = 1+6+8$$
$$15_d = 13_z = 3+4+8 = 3+5+7$$
$$15_d = 13_z = 7+2+6 = 7+3+5$$
$$15_d = 13_z = 9+1+4 = 9+2+4$$

Hence if we place 3 in the (1,2) position, then we must place 7 in the (3,2) position. Finally if we place 9 in the (2,1) position, we must place 1 in the the (2,3) position. This completes the magic square.

With the above information, we immerse in the following activities:

1. First, we rotate the magic square above 90°_d clockwise and 90°_d counterclockwise. We next rotate the magic square above both 180°_d clockwise and 180°_d counterclockwise. Finally we illustrate the magic squares obtained and ask if they are really different magic squares.

2. In addition, we add 10_d = Ƨ$_z$ to each entry in the original magic square and generate some conclusions.

3. Our final activity in this section involves multiplying each entry in the original magic square by 5 and form some conclusions.

Solutions to the Above Immersions

1. If we rotate the original magic square

4	3	8
9	5	1
2	7	6

90°_d clockwise, we obtain the following magic square:

2	9	4
7	5	3
6	1	8

If we rotate the original magic square 90°_{d} counterclockwise, we obtain the following magic square:

8	1	6
3	5	7
4	9	2

If we rotate the original magic square 180°_{d} counterclockwise, we obtain the following magic square:

6	7	2
1	5	9
8	3	4

These magic squares are not different from the original one as rotations preserve the magic sum.

2. If we add 10_{d}=$↊_{\text{z}}$ to each entry in the original magic square, we still obtain a magic square with magic sum 45_{d}=39_{z}:

[d] all decimal

14	13	18
19	15	11
12	17	16

≡

[z] all dozenal

12	11	16
17	13	↋
10	15	14

3. If we multiply each entry in the original magic square by 5, we likewise obtain a magic square whose magic sum is five times that of the original magic square and hence 75_{d}=63_{z}:

[d] all decimal

20	15	40
45	25	5
10	35	30

≡

[z] all dozenal

18	13	34
39	31	5
↊	2↋	26

It is easy to see that if one adds the same constant to each entry in a magic square, the resulting configuration will likewise result in a magic square. To justify this in the (3) case, consider the following magic square with generic entries a, b, c, d, e, f, g, h, i in the configuration below:

a	b	c
d	e	f
g	h	i

Suppose the sum of the entries in every row, column and both diagonals in this magic square is k. In the above configuration, we arrive at the following eight equations:

$$
\begin{array}{lll}
a+b+c=k & d+e+f=k & g+h+i=k \\
a+d+g=k & b+e+h=k & c+f+i=k \\
a+e+i=k & g+e+c=k &
\end{array}
$$

Suppose we add m to each of the entries, we obtain the following magic square with magic sum $k+3m$.

$a+m$	$b+m$	$c+m$
$d+m$	$e+m$	$f+m$
$g+m$	$h+m$	$i+m$

Rudolf Ordrejka's Magic Square

We first show that the following (3×3) configuration is a magic square consisting of nine primes (in red) attributed to Rudolf Ordrejka $(1928\text{-}2001)_{\mathrm{d}}$ $(1148\text{-}11↊9)_{\mathrm{z}}$ with magic sum $177_{\mathrm{d}}=129_{\mathrm{z}}$:

[d] all decimal

17	89	71
113	59	5
47	29	101

≡

[z] all dozenal

15	75	5↋
95	4↋	5
3↋	25	85

Observe the following:

[d] all decimal	[z] all dozenal
17 + 89 + 71 = 177	15 + 75 + 5↋ = 129
113 + 59 + 5 = 177	95 + 4↋ + 5 = 129
47 + 29 + 101 = 177	3↋ + 25 + 85 = 129
17 + 113 + 47 = 177	15 + 95 + 3↋ = 129
89 + 59 + 29 = 177	75 + 4↋ + 25 = 129
71 + 5 + 101 = 177	5↋ + 5 + 85 = 129
17 + 59 + 101 = 177	15 + 4↋ + 85 = 129
47 + 59 + 71 = 177	3↋ + 4↋ + 5↋ = 129

Decimal Case: Note that the units' digits for primes larger than 5 in decimal must necessarily end in one of the digits 1, 3, 7 or 9. Hence adding the constant $\ldots 0_{\mathrm{d}}$ (i.e., any integer with any number of decimal digits ending in 0) to every term in the magic square would have the entry in the (2,3) position ending in the digit 5 and hence not prime. Adding the constant $\ldots 2_{\mathrm{d}}$ to every term in the magic square would force the entry in the (2,1) position to terminate in the digit 5. Adding the constant $\ldots 4_{\mathrm{d}}$ to each term in the magic square would force the entries in the (1,3) and (3,3) positions to end in the digit 5; while adding the constant $\ldots 6_{\mathrm{d}}$ to each of the entries in the second column would yield the resulting entries terminating in the digit 5. Similarly, adding the constant $\ldots 8_\text{d}$ forces the entries in the (1,1) and (3,1) positions to terminate in the digit 5. Since all the entries in the magic square of primes are odd, adding any integer terminating in an odd digit makes the resulting integer even and thus not prime. Hence it is impossible to add any constant to the magic square to produce a new magic square such that all nine entries are primes.

Dozenal Case: Note that the units' digit of odd primes can end in any one of the dozenal digits 1, 5, 7 or Ɛ. On the other hand, observe that adding the constant $\ldots 4_\text{z}$ (i.e., an integer with any number of dozenal digits ending in 4) to every term in the original magic square would force the entries in the (1,3), (2,2), *and* (3,1) positions to terminate in the digit 3 while the entries in each of the other positions to terminate in the digit 9. Thus none of the entries in the new magic square would be prime.

On the other hand, there is the potential to have up to eight of the entries being prime when a positive even integer constant is added to each of the entries in the given magic square of primes. We next cite some examples below with the prime entries in Red.

It should be noted that adding any positive odd integer to Rudolph Ordrejka's original magic square would force all of the resulting entries even and larger than two and hence not prime. While infinitely many solutions are generated in this manner, the solution is neither interesting nor elegant.

On the other hand, adding 4 to each entry produces a magic square having no primes and magic sum 189_d=139_z:

[d] all decimal

21	93	75
117	63	9
51	33	105

≡

[z] all dozenal

19	79	63
99	53	9
43	29	89

Each of the entries in this new magic square is divisible by 3. Recall that any dozenal numeral ending in the digits 0, 3, 6 or 9 is divisible by 3.

Moreover if one adds any integer of the form $6n-2$ ($n \in \mathbb{N}$) to every entry in Rudolf Ordrejka's magic square, one produces a magic square consisting of no primes; for every element in the magic square will be divisible by 3, as evidenced by the digit root decimal, or the final digit in dozenal. Hence adding 4, 10_d=$Ϩ_\text{z}$, 16_d=14_z, etc. to each entry of the original magic square will produce a magic square consisting of no prime entries. Adding 4 to each entry of Rudolf Ordrejka's magic square produces this result for the initial time.

One related observation is to note that every integer in the original magic square has a remainder of two upon division by three. Using the language of congruences in mathematical parlance, we assert that each entry is congruent to two modulo three. In general, we say that a is congruent to b modulo n, written $a \equiv b \pmod{n}$ if and only if a and b have the same remainder upon division by n, or equivalently, $n|(a-b)$.

Hence we observe that $17_\text{d} = 15_\text{z} \equiv 2 \pmod 3$; $3|(17_\text{d} - 2 = 15_\text{z} - 2) \Leftrightarrow 3|(15_\text{d} = 13_\text{z})$ (since of course $15_\text{d} = 13_\text{z} = 3 \cdot 5$). The other entries are verified similarly.

We next focus on securing the smallest constants one needs to add to each entry in Ordrejka's prime magic square to produce dozenal magic squares consisting of from

0-8 prime entries inclusive. In each case, we are starting with Rudolph Ordrejka's original magic square, produced below for ready reference.

[d] all decimal				[z] all dozenal		
17	89	71		15	75	5↋
113	59	5	≡	95	4↋	5
47	29	101		3↋	25	85

Adding $666_{\mathrm{d}}=476_{\mathrm{z}}$ to each entry produces a magic square having one prime for the first time with magic sum $2175_{\mathrm{d}}=1313_{\mathrm{z}}$:

[d] all decimal				[z] all dozenal		
683	755	737		48↋	52↋	515
779	725	671	≡	54↋	505	4↋5
713	695	767		47↋	4↋5	49↋

Adding $116_{\mathrm{d}}=98_{\mathrm{z}}$ to each entry produces a magic square having two primes for the first time with magic sum $525_{\mathrm{d}}=379_{\mathrm{z}}$:

[d] all decimal				[z] all dozenal		
133	205	187		↋1	151	137
229	175	121	≡	171	127	↊1
163	145	217		117	101	161

Adding $74_{\mathrm{d}}=62_{\mathrm{z}}$ to each entry produces a magic square having three primes for the first time with magic sum $399_{\mathrm{d}}=293_{\mathrm{z}}$:

[d] all decimal				[z] all dozenal		
91	163	145		77	117	101
187	133	79	≡	137	↋1	67
121	103	175		↊1	87	127

Adding $32_{\mathrm{d}}=28_{\mathrm{z}}$ to each entry produces a magic square having four primes for the first time with magic sum $273_{\mathrm{d}}=1↊9_{\mathrm{z}}$:

[d] all decimal				[z] all dozenal		
49	121	103		41	↊1	87
145	91	37	≡	101	77	31
79	61	133		67	51	↋1

Adding $18_{\mathrm{d}}=16_{\mathrm{z}}$ to each entry produces a magic square having five primes for the first time with magic sum $231_{\mathrm{d}}=173_{\mathrm{z}}$:

[d] all decimal				[z] all dozenal		
35	107	89		2↋	8↋	75
131	77	23	≡	↊↋	65	1↋
65	47	119		55	3↋	9↋

Adding 2 to each entry produces a magic square having six primes for the first time with magic sum 183_{d}=133_{z}:

[d] all decimal

19	91	73
115	61	7
49	31	103

≡

[z] all dozenal

17	77	61
97	51	7
41	27	87

Adding 14_{d}=12_{z} to each entry produces a magic square having seven primes for the first time with magic sum 219_{d}=163_{z}:

[d] all decimal

31	103	85
127	73	19
61	43	115

≡

[z] all dozenal

27	87	71
↊7	61	17
51	37	97

Adding 12_{d}=10_{z} to each entry produces a magic square having eight primes for the first time with magic sum 213_{d}=159_{z}:

[d] all decimal

29	101	83
125	71	17
59	41	113

≡

[z] all dozenal

25	85	6↋
↊5	5↋	15
4↋	35	95

Note that such answers are not unique and the reader can secure multiple solutions! For example, adding 26_{d}=22_{z} to each entry in Rudolf Ordrejka's magic square likewise produces a magic square having 6 primes with magic sum 255_{d}=193_{z}, namely:

[d] all decimal

43	115	97
139	85	31
73	55	127

≡

[z] all dozenal

37	97	81
↋7	71	27
61	47	↊7

Additional Observations

It was earlier mentioned that the magic squares produced by adding any integer of the form $6n - 2$ ($n \in \mathbb{N}$) to each entry of Rudolf Ordrejka's magic square, one produces a magic square consisting of no primes; for every element in the magic square will be divisible by three by the sum of the digits test. The reason for this follows from the following congruence (for we are adding to elements congruent to two modulo three elements that are congruent to one modulo three so that the sum is congruent to zero modulo three):

If $a \equiv b (\mathrm{mod}\ n)$ and $c \equiv d (\mathrm{mod}\ n)$, then $a + c \equiv b + d (\mathrm{mod}\ n)$. (1)

For example:

[d] all decimal	[z] all dozenal
$17 \equiv 2 \pmod 3 \Leftrightarrow$	$15 \equiv 2 \pmod 3 \Leftrightarrow$
$17 + 4 \equiv (2 + 4) \pmod 3 \Leftrightarrow$	$15 + 4 \equiv (2 + 4) \pmod 3 \Leftrightarrow$
$21 \equiv 6 \pmod 3 \Leftrightarrow$	$19 \equiv 6 \pmod 3 \Leftrightarrow$
$3 \mid (21 - 6) \Leftrightarrow$	$3 \mid (19 - 6) \Leftrightarrow$
$3 \mid 15 \Leftrightarrow$	$3 \mid 13 \Leftrightarrow$
$15 = 3 \cdot 5$	$13 = 3 \cdot 5$

To prove (1), note that:

$$\text{Since } a \equiv b \pmod n, \qquad n \mid (a - b) \qquad (2)$$
$$\text{Since } c \equiv d \pmod n, \qquad n \mid (c - d) \qquad (3)$$

Now using elementary properties of divisibility, since $n \mid (a - b)$ and $n \mid (c - d)$,

$$\begin{aligned} &n \mid [(a - b) + (c - d)] \Leftrightarrow \\ &n \mid [a - b + c - d] \Leftrightarrow \\ &n \mid [a + c - b - d] \Leftrightarrow \\ &n \mid [(a + c) - (b + d)] \Leftrightarrow \\ &a + c \equiv (b + d) \pmod n \end{aligned}$$

An additional question was raised with regards to the above. Is it possible to find a constant to add to Rudolph Ordrejka's original magic square to obtain one consisting of no primes, but none of the entries is divisible by three? The possibility for this certainly exists; for as one proceeds further out in the set of positive primes (which is infinite and first proven by Euclid over two thousand years ago), the gap between consecutive primes can be made as wide as one pleases by considering the following argument in which we seek to secure n consecutive composite integers:

Consider the sequence

$$\{(n + 1)! + 2, (n + 1)! + 3, (n + 1)! + 4, \ldots, (n + 1)! + (n + 1)\}$$

The first term is divisible by 2, the second term is divisible by 3, the third term is divisible by 4 and so forth while the last term is divisible by $n + 1$. Hence one has n consecutive composite integers. This sequence is definitely not minimal.

For example, to secure a sequence consisting of five consecutive integers utilizing this constructive process, we form the sequence:

$$\begin{aligned} (5 + 1)! + 2 = 6! + 2 = 720_{\mathrm{d}} + 2 = 722_{\mathrm{d}} = 500_{\mathrm{z}} + 2 = 502_{\mathrm{z}} \\ (5 + 1)! + 3 = 6! + 3 = 720_{\mathrm{d}} + 3 = 723_{\mathrm{d}} = 500_{\mathrm{z}} + 3 = 503_{\mathrm{z}} \\ (5 + 1)! + 4 = 6! + 4 = 720_{\mathrm{d}} + 4 = 724_{\mathrm{d}} = 500_{\mathrm{z}} + 4 = 504_{\mathrm{z}} \\ (5 + 1)! + 5 = 6! + 5 = 720_{\mathrm{d}} + 5 = 725_{\mathrm{d}} = 500_{\mathrm{z}} + 5 = 505_{\mathrm{z}} \\ (5 + 1)! + 6 = 6! + 6 = 720_{\mathrm{d}} + 6 = 726_{\mathrm{d}} = 500_{\mathrm{z}} + 6 = 506_{\mathrm{z}} \end{aligned}$$

The first term is divisible by 2, the second term is divisible by 3, the third term is divisible by 4, the fourth term is divisible by 5 and the fifth term is divisible by 6. Of course the minimal sequence of five consecutive composite integers is far smaller, namely $\{24_{\mathrm{d}}, 25_{\mathrm{d}}, 26_{\mathrm{d}}, 27_{\mathrm{d}}, 28_{\mathrm{d}}\} \equiv \{20_{\mathrm{z}}, 21_{\mathrm{z}}, 22_{\mathrm{z}}, 23_{\mathrm{z}}, 24_{\mathrm{z}}\}$.

If one reverts to the previous question, I noted that the difference between the largest and smallest integers in the original magic square is $113_{\mathrm{d}} - 5 = 108_{\mathrm{d}} = 95_{\mathrm{z}} - 5 = 90_{\mathrm{z}}$.

Hence I looked to secure a prime gap of this size. The integer 2238823_{d}=8ƐƐ747$_{\text{z}}$ is prime and there is a gap of 108_{d}=90_{z} for the first time following this prime. The next prime is 2238931_{d}=8ƐƐ817$_{\text{z}}$. Hence the integers from 2238824_{d}=8ƐƐ748$_{\text{z}}$ to 2238930_{d}=8ƐƐ816$_{\text{z}}$ are all composite.

If we add the constant 2238822_{d}=8ƐƐ746$_{\text{z}}$ to all terms of the original magic square, we obtain a magic square consisting of no primes with none of the entries divisible by three.

Is there a minimum constant that achieves our goal? Indeed the answer to this question is in the affirmative and a MATHEMATICA Program was run to accomplish this. It turns out that the constants must be congruent to either zero modulo three or two modulo three so that the resulting sums with the entries (all congruent to two modulo three in the original magic square) are congruent to either two modulo three or one modulo three respectively and hence are not divisible by three. The constants 1656_{d} = Ɛ60$_{\text{z}}$ $\equiv 0 \pmod 3$ and 2876_{d} = 17Ɛ8$_{\text{z}}$ $\equiv 2 \pmod 3$ are optimal.

Conclusion

This article served to furnish some engaging mathematics dealing with magic squares that is accessible and hopefully fun mathematics for students and teachers from a wide spectrum of grade levels ranging from elementary through college and beyond. In addition, the posing of thoughtful questions is key for a productive teaching and learning experience. With regards to *The Standards For Mathematical Practice* articulated in The Common Core, this article attempted to demonstrate the necessity to make sense of problems and persevere in solving them, make use of structure, use appropriate tools strategically, construct viable arguments and critique the reasoning of others and express regularity in repeated reasoning (modular arithmetic). One might argue with regards to the other practices as well as the related NCTM works of Peg Smith and Mary Kay Stein. The references below will provide the reader with additional magic squares to partake in as well as related matters. The excellent websites Mathworld and The On-Line Encyclopedia of Integer Sequences can further excite, invigorate and truly fuel one's passion for mathematics. I invite the reader to further partake of these ideas and always enjoy the mathematical journey!

References

1. *MathWorld - A Wolfram Resource*, Champaign-Urbana, IL (2018_{d}) (1202_{z}).

2. *Principles and Standards for School Mathematics*, The National Council of teachers of Mathematics (NCTM), Reston, VA (1990_{d}) (119↊$_{\text{z}}$).

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4. Smith, P. and Stein, M.K., *5 Practices for Orchestrating Productive Mathematical Discussions*, Corwin Press, NCTM (2011_{d}) (11Ɛ7$_{\text{z}}$).

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Pounds, Shillings, Pence

The Commonwealth's Dozenal Currency

by Andon Epp

Currency is not the first application of dozenal that one may think of. Measurement in dozenal has been well explored, as there are many dozens already present in some unit relationships (20_z hours/day, 10_z months/year, 10_z inches/foot, and more), and several attempts to create a coherent dozenal metric system have been devised, some of which were explored in *Bulletin* issue ↊3_z.[1]

Global systems of money, however, have been mostly decimal for a biquennium or two. Starting with the Russian ruble and US dollar, the division of basic units of currency into 100_d parts gradually became standard throughout the $11st_z$ and $12nd_z$ biquennia ($18th_d$ through $20th_d$ centuries). Today, with these subdivisions ingrained into digital bank accounts and credit card data, the chance appears slim that they will be changed, although some countries have eliminated their low-value physical coins. However, some non-decimal systems of money lingered on, and the most notable and interesting for dozenalists was pounds, shillings, and pence (£sd), which predominated in Britain and Commonwealth nations.

The system consisted of one dozen (12_d) pence to a shilling, and one dozen eight (20_d) shillings to a pound. Combinations of amounts were written and spoken in these three units: for example, as three shillings and fivepence or "three and five" (3s 5d or 3/5), or two pounds, thirteen shillings and sevenpence (£2/13/7_d). It originated from ancient Roman accounting,[2] which at first recognized the bronze *as* and silver *denarius*, originally in a ten-to-one relationship; later the *solidus* and *libra* were added. The relationships between these units were inconsistent among various regions until the eighth century (sixth biquennium), when the French emperor Charlemagne declared that the *libra* would be valued at a troy pound of silver, worth 20_d *solidi* or 240_d *denarii* (silver pennyweights). This standardization soon spread through continental Europe; it became known as *livres-sous-deniers* in France, and England (along with the rest of Britain) adopted them as pounds-shillings-pence after the Norman conquest. Note that the same abbreviations, £sd, were used for these, regardless of language.

The UK minted coins of various denominations during the use of £sd, including fractions of pence. Table 1[3] [4] [5] shows these coins with their values and mintage dates. The dozen-to-one ratio of the shilling and penny made the lower denominations well suited for dozenal numeration.[6] The circulating coins from farthing to sixpence come out as simple dozenal radix expressions and fractions of a shilling. The system carried with it the benefits of the dozen's divisibility; a shilling could be split

[1] https://dozenal.org/drupal/sites_bck/default/files/DuodecimalBulletinIssue531.pdf

[2] https://en.wikipedia.org/wiki/%C2%A3sd

[3] https://en.wikipedia.org/wiki/List_of_British_banknotes_and_coins

[4] The "halfpenny" was more commonly contracted to "ha'penny."

[5] The "crown" was used mostly for commemorative purposes.

[6] Covered in A.C. Aitken's "The Case Against Decimalisation," which also entertained the idea of a "royal," a dozen-shilling unit. http://dozenalsociety.org.uk/pdfs/aitken.pdf

Table 1. Values and Mintage Eras of Various £sd Coins

Name	Value [d]	Value [z]	Mintage [d]	Mintage [z]
Quarter farthing	1/16 d	0.009 s	1839–1868	1093–10Ɛ8
Third farthing	1/12 d	0.01 s	1827–1913	1083–1135
Half farthing	1/8 d	0.016 s	1828–1856	1084–10↊8
Farthing	1/4 d	0.03 s	c.1200–1960	c.840–1174
Halfpenny	1/2 d	0.06 s	1260–1969	890–1181
Penny	1 d	0.1 s	757–1970	531–1182
Thruppenny bit	3 d	0.3 (1/4) s	1547–1970	↊8Ɛ–1182
Groat	4 d	0.4 (1/3) s	1836–1862	1090–10Ɛ2
Sixpence	6 d	0.6 (1/2) s	1547–1970	↊8Ɛ–1182
Shilling	1 s (12 d)	1 s	1502–1970	↊52–1182
Florin	2 s	2 s	1848–1970	10↊0–1182
Half crown	2 s 6 d	2.6 s	1528–1969	↊74–1181
Double florin	4 s	4 s	1887–1890	1113–1116
Crown	5 s	5 s	1528–1969	↊74–1181
Half sovereign	10 s	↊ s	1817–1937	1075–1155
Sovereign	£1 (20 s)	18 s	1817–1937	1075–1155

into 2, 3, 4, 6, and even 8 if halfpence were used. The "pence table" learned in schools simply employed dozenal multiplication, with several repeating patterns. There were also slight mental arithmetic advantages: if the price of a dozen items was 3/– (three shillings), each item clearly cost 3d each.

The pound's value of twenty shillings made £sd a hybrid system, which made working large sums of money more difficult, and it became the main reason the system was abandoned. The British half-crown (2/6), at 1/8 of a pound, functioned much like the US quarter, and its value was even converted to 25_{d}¢ in some countries during decimalization. The double florin (1/5 of £1) may have accelerated decimal currency had it caught on with the public, but it was only minted for four years. There was even a short-lived "guinea" series based on multiples/fractions of 19_{z} (21_{d}) shillings, but the coins were unpopular, although guineas were still used as value terms. In fact, since prices and amounts above around £5 were uncommon prior to post-decimal inflation in the 1180s_{z} (1970s_{d}), the dozenal portion of £sd predominated in practice.

The UK and Commonwealth began debating decimal conversion of money in the 1170s_{z} (1960s_{d}), with each country soon making the switch in conjunction with its adoption of metric units. Most, such as South Africa, Australia, and New Zealand, chose to make ten former shillings the new unit, dividing it into 100_{d} cents. This led to a shilling worth 10_{d}¢ and a new 1¢ coin close in value to the old penny. The British government elected to keep the pound, making a shilling 5 "new pence" (100_{d} p making £1). Coins from the sixpence upward could still be used in transactions at their decimal values after the conversion (see Table 2), but they have since ceased to be legal tender.

Indeed, £sd was not strictly *dozenal*, as it also had a strong vigesimal element, and all the multiples of each denomination were expressed in decimal anyway. Consequently,

Table 2. Conversion of £sd coins to decimal.

Coin	Abbr [d]	Value (UK) in £p [d]	Value (Aus, NZ) in $¢ [d]
Penny	1d	Approx. ½p	Approx. 1¢
Thruppenny bit	3d	Approx. 1p	Approx. 2¢
Sixpence	6d	2½p	5¢
Shilling	1/–	5p	10¢
Florin	2/–	10p	20¢
Crown	5/–	25p	50¢
Ten-shilling note	10/–	50p (coin)	$1
Pound note	£1	£1 (note)	$2

on the DozensOnline forum,[7] some members of the DSA and DSGB have pondered what true dozenal money could look like in principle.

In a thread titled "What Unit Would You Use in a New Dozenal Currency,"[8] user "Dan" proposes retaining the main unit (be it the dollar, pound, euro, or anything else) and divide it into uncias. He notes that this would preserve America's half-dollars and quarters. However, a bicia·dollar would be worth less than the penny, which would be problematic since the penny is already being considered for discontinuation.

In the same thread, user "Joshbuckler" proposes retaining the penny as is, but building up a new main currency unit out of a gross of pennies. User "Kodegadulo" echoes this in the thread "A Problem and a Solution to Converting Decimal to Dozenal Currency."[9] This would avoid the issue of rounding error when converting from decimal currency to dozenal. Given how much software there is these days which is keeping track of financial accounts precisely to the penny, such rounding errors could be especially disruptive. That said, replacing a country's main currency unit with something else runs the risk of stripping its currency of its international prestige.[↊]

See Table 3 to compare and contrast these proposals. (Bicia·dollars are included for comparison, but are shown in gray.)

Proposals for coins also differ. The most popular solution is a $1\text{-}3\text{-}6\text{-}10\text{-}30\text{-}60_z$ system. This would be analogous to Europe's $1\text{-}2\text{-}5\text{-}10\text{-}20\text{-}50_d$ system, but with the added benefit of each denomination being a multiple of the previous one (see Table 3). However, other configurations, including $1\text{-}3\text{-}10_z$ and $1\text{-}4\text{-}6\text{-}10_z$, are also viable.

Regularizing currency around twelves alone would make for clean divisibility throughout the system. For instance, dividing any dozenal power by 3 gives 4 of the next power down, and this works at all powers:

$$\begin{aligned}
\$0.10_z \div 3 &= \$0.04_z \\
\$1.00_z \div 3 &= \$0.40_z \\
\$10.00_z \div 3 &= \$4.00_z \\
\$100.00_z \div 3 &= \$40.00_z
\end{aligned}$$

[7] https://www.tapatalk.com/groups/dozensonline/currency-systems-f20/

[8] https://www.tapatalk.com/groups/dozensonline/viewtopic.php?t=598

[9] https://www.tapatalk.com/groups/dozensonline/viewtopic.php?t=2148

[↊] Kodegadulo himself may not be taking this proposal completely seriously, because he whimsically suggests calling the new biqua·penny unit a U.S. "zachary" (with abbreviation Ƶ) in honor of Zachary Taylor. Zachary Taylor happens to be the *dozenth* president of the United States—but not a particularly memorable one, since he died after only a year in office, apparently of indigestion.

Kodegadulo also suggests calling the quadqua·penny or biqua·zachary (Ƶ100.00_z) a "woodrow", after Woodrow Wilson. Wilson happened to be president on July 4, 1140_z (1920_d), the first biquennial anniversary of the Declaration of Independence. However, given that he spent much of that year incapacitated by a stroke, it was not the most auspicious celebration.

Contrast this with £sd which carried the dozen in only one position. A third of a shilling was simply 4 pence, but a third of a pound was 6 shillings 8 pence.

$$\begin{aligned} 1\text{s} \div 3 &= \phantom{6\text{s }}4\text{d} \\ £1 \div 3 &= 6\text{s } 8\text{d} \end{aligned}$$

In a decimal world, the pragmatic choice was to enhance decimal with auxiliary factors of three, because they are missing in decimal yet useful enough to warrant inclusion. This can be seen in cases such as sexagesimal time, US customary measures, the ISO 2848 standard for metric building construction,[Ɛ] and of course £sd currency. But in a dozenal world, with fully dozenal measurement and timekeeping, as well as fully dozenal currency, would there be an analogous choice to enhance dozenal with auxiliary factors of five? Such a world would likely feel no need to augment the already-rich divisors of dozenal; a factor of five might seem almost as alien as seven. So the inconveniences of independent arithmetic methods could disappear.

When the United Kingdom completed its program with Decimal Day on February 13_z, 1183_z (February 15_d, 1971_d)[10] a long-lasting and interesting currency system was discontinued. The historical significance of pounds, shillings, and pence, however, has inspired some dozenalists to go all in on using their favorite number to divvy up the bill. ⁞⁞

Table 3. Potential Conversions of Decimal Money to Dozenal

Retain the Dollar and Divide It Dozenally		Retain the Penny Propose Biqua·Penny as Main Unit	
Coin/Note Value [z]	Equivalent Dollars [d]	Coin/Note Value [z]	Equivalent Dollars [d]
\$0.01	\$0.0069$\overline{4}$	Ƶ0.01	\$0.01
\$0.03	\$0.0208$\overline{3}$	Ƶ0.03	\$0.03
\$0.06	\$0.041$\overline{6}$	Ƶ0.06	\$0.06
\$0.10	\$0.08$\overline{3}$	Ƶ0.10	\$0.12
\$0.30	\$0.25	Ƶ0.30	\$0.36
\$0.60	\$0.50	Ƶ0.60	\$0.72
\$1.00	\$1.00	Ƶ1.00	\$1.44
\$3.00	\$3.00	Ƶ3.00	\$4.32
\$6.00	\$6.00	Ƶ6.00	\$8.64
\$10.00	\$12.00	Ƶ10.00	\$17.28
\$30.00	\$36.00	Ƶ30.00	\$51.84
\$60.00	\$72.00	Ƶ60.00	\$103.68
\$100.00	\$144.00	Ƶ100.00	\$207.36

[Ɛ] Which uses "preferred numbers" that include a factor of 3, to introduce 300_d mm ("metric foot"), 600_d mm, and 1200_d mm units. See https://en.wikipedia.org/wiki/ISO_2848

[10] For a retrospective on the process of decimalization in the UK, see the "Funny Money" video series: (1) https://youtu.be/ljeW5LHzNKg, (2) https://youtu.be/4tnn_8TXlEc, (3) https://youtu.be/pCOf2uP26wk.

POETRY CORNER

Multiverse Multi-Verse

Could there be parallel worlds where English-speaking people (including poets) tally their numbers in twelves or eights or twenties, as the case may be? Here's a well-known rhyme by an anonymous Elizabethan bard (not!) from a tantalizing triptych of alternate timelines:

Dozenal-Verse

Twenzy-six days hath September,
April, June, and November.
Twenzy-seven for the rest,
Save Febru'ry doth fail this test,
Having only twenzy-four,
Tho' it once possess'd two more.
Whither they went hath long been
reckon'd:
July stole one, August the second.
But when a leap year doth arrive,
'Twill gain one back, for twenzy-five.
Every four years mark that date,
Save once in ev'ry tenzy-eight.
Trimming thus the septdubquennia,
Will keep us, for a few triquennia,
Tracking ev'ry passing season
With perfect rhythm, rhyme & reason.

Octal-Verse

Throcty-six days hath September,
April, June, and November.
Throcty-seven for the rest,
Save Febru'ry doth fail this test,
Having only throcty-four,
Tho' it once possess'd two more.
Whither they went hath long been
reckon'd:
July stole one, August the second.
But when a leap year doth arrive,
'Twill gain one back, for throcty-five,
Quadrennially earning such reward,
Save once each second checkerboard.
Trimming thus the septdubquennia,
Will keep us, for a few triquennia,
Tracking ev'ry passing season
With perfect rhythm, rhyme & reason.

Vigesimal-Verse

Score & ten days hath September,
April, June, and November.
Score & 'leven for the rest,
Save Febru'ry doth fail this test,
Having only eight & score,
Tho' it once possess'd two more.
Whither they went hath long been
reckon'd:
July stole one, August the second.
But when a leap year's stars align,
'Twill gain one back, for score & nine.
Every four years mark that date,
Save once in each six-score & eight.
Trimming thus the septdubquennia,
Will keep us, for a few triquennia,
Tracking ev'ry passing season
With perfect rhythm, rhyme & reason.

All of these verses imply the same leap-year calculation based on powers of 2, shown below.[1] [2] This yields an average year precisely equal to the nominal tropical year of three hundred sixty-five days, five hours, forty-eight minutes, and forty-five seconds; or two gross sixzy-five days, two dwells, ten breathers, and ten and a half trices; or five checkerboard, fivocty-five days and octy-seven-and-a-half percheckerboard; or eighteen-score-and-five days and four perscore, sixteen perscoresquare and seventeen-and-a-half perscore-cube. Despite involving no round centuries or biquennia, this scheme more accurately tracks the seasons than the Gregorian calendar average of 365.2425_d days. "Unleap" years, applying the rule proleptically, are shown at bottom. (Bases indicated by color. Latin uppercase transdecimals used for consistency.)

Decimal

30	×	4	=	120
31	×	7	=	217
28	×	1	=	28
1	÷	4	=	0.25
−1	÷	128	=	−0.0078125
				365.2421875

Dozenal

26	×	4	=	A0
27	×	7	=	161
24	×	1	=	24
1	÷	4	=	0.3
−1	÷	A8	=	−0.0116
				265.2AA6

Octal

36	×	4	=	170
37	×	7	=	331
34	×	1	=	34
1	÷	4	=	0.2
−1	÷	200	=	−0.004
				555.174

Vigesimal

1A	×	4	=	60
1B	×	7	=	AH
18	×	1	=	18
1	÷	4	=	0.5
−1	÷	68	=	−0.032A
				I5.4GHA

Days	hh:mm:ss
2^{-2} =	06:00:00
-2^{-7} =	−00:11:15
	05:48:45

Days	hh:mm:ss
$2 \times \mathtt{C}^{-1}$ =	04:00:00
$\mathtt{A} \times \mathtt{C}^{-2}$ =	01:40:00
$\mathtt{A} \times \mathtt{C}^{-3}$ =	00:08:20
$6 \times \mathtt{C}^{-4}$ =	00:00:25
	05:48:45

Days	hh:mm:ss
1×8^{-1} =	03:00:00
7×8^{-2} =	02:37:30
4×8^{-3} =	00:11:15
	05:48:45

Days	hh:mm:ss.s
$4 \times \mathtt{K}^{-1}$ =	04:48:00.0
$\mathtt{G} \times \mathtt{K}^{-2}$ =	00:57:36.0
$\mathtt{H} \times \mathtt{K}^{-3}$ =	00:03:03.6
$\mathtt{A} \times \mathtt{K}^{-4}$ =	00:00:05.4
	05:48:45.0

Unleap Years (decimal)

128	640	1152	1664
256	768	1280	1792
384	896	1408	1920
512	1024	1536	2048

Unleap Years (dozenal)

A8	454	800	B68
194	540	8A8	1054
280	628	994	1140
368	714	A80	1228

Unleap Years (octal)

200	1200	2200	3200
400	1400	2400	3400
600	1600	2600	3600
1000	2000	3000	4000

Unleap Years (vigesimal)

68	1C0	2HC	434
CG	1I8	340	49C
J4	24G	3A8	4G0
15C	2B4	3GG	528

[1] In Systematic Numeric Nomenclature (SNN) generalized to multiple bases, *septdubquennium* means 2^7 years.
[2] Historically, February did not directly lose days to July and August—but it's a fun conceit.

The Dozenal Society of America

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